

GW Invariants Relative Normal Crossings Divisors

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Abstract

In this paper we introduce a notion of symplectic normal crossings divisor V and define the GW invariant of a symplectic manifold X relative such a divisor. Our definition includes normal crossings divisors from algebraic geometry. The invariants we define in this paper are key ingredients in symplectic sum type formulas for GW invariants, and extend those defined in our previous joint work with T.H. Parker [IP1], which covered the case V was smooth. The main step is the construction of a compact moduli space of relatively stable maps into the pair (X, V) in the case V is a symplectic normal crossings divisor in X .

0 Introduction

In previous work with Thomas H. Parker [IP1] we constructed the relative Gromov-Witten invariant $GW(X, V)$ of a closed symplectic manifold X relative a smooth “divisor” V , that is, a (real) codimension 2 symplectic submanifold. These relative invariants are defined by choosing an almost complex structure J on X that is compatible with both V and the symplectic form, and counting J -holomorphic maps that intersect V with specified multiplicities. An important application is the symplectic sum formula that relates the GW invariant of a symplectic sum $X \#_V Y$ to the relative GW invariants of (X, V) and (Y, V) (see [IP2] and the independent approaches [LR], [Li] and [EGH]).

In this paper we introduce a notion of symplectic normal crossings divisor V and define the GW invariant of a symplectic manifold X relative such a divisor. Roughly speaking, a set $V \subset X$ is a symplectic normal crossings divisor if it is locally the transverse intersection of codimension 2 symplectic submanifolds compatible with J (the precise definition is given in Section 1).

There are many reasons why one would want to extend the definition of relative GW invariants to include normal crossings divisors, and we already have several interesting applications in mind. One is a Mayer-Vietoris type formula for the GW invariants: a formula describing how the GW invariants behave when X degenerates into several components and that allows one to recover the invariants of X from those of the components of the limit. The simplest such degenerations come from the symplectic sum along a smooth divisor. But if one wants to iterate this degeneration, one is immediately confronted with several pieces whose intersection is no longer smooth, but instead are normal crossings divisors. Normal crossings divisors appear frequently in algebraic geometry, not only as the central fiber of a stable degeneration but also for example as the toric divisor in a toric manifold which then

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appears in the context of mirror symmetry. We also have some purely symplectic applications in mind in which normal crossings divisors arise from Donaldson's theorem; these will appear in a subsequent paper.

The general approach in this paper is to appropriately adapt the ideas in [IP1] but now allow the divisor to have a simple type of singularity, which we call symplectic normal crossings. This is defined in Section 1, where we also present many of the motivating examples. The notion of simple singularity is of course relative: the main issue here is to be able to control the analysis of the problem; the topology of the problem, though perhaps much more complicated is essentially of a combinatorial nature so it is much easier controlled.

There are several new features and problems that appear when the divisor V has such singular locus. First, one must include in the moduli space holomorphic curves that intersect the singular locus, and one must properly record the contact information about such intersections. In Section 2 we describe how to do this and construct the corresponding moduli space $\mathcal{M}_s(X, V)$ of stable maps into X whose contract intersection with V is described by the sequence s . There is a lot of combinatorics lurking in the background that keeps track of the necessary topological information along the singular locus, which could make the paper unnecessarily longer. We have decided to keep the notation throughout the paper to a minimum, and expand its layers only as needed for accuracy in each section. We give simple examples of why certain situations have to be considered, explain in that simple example what needs to be done, and only after that proceed to describe how such situations can be handled in general. In the Appendix we describe various needed stratifications associated to a normal crossings divisor, and topological data associated to it.

The other more serious problem concerns the construction of a compactification $\overline{\mathcal{M}}_s(X, V)$ of the relative moduli space. In the usual Gromov compactification of stable maps into X , a sequence of holomorphic maps that have a prescribed contact to V may limit to a map that has components in V or even worse in the singular locus of V ; then not only the contact information is lost in the limit, but the formal dimension of the corresponding boundary stratum of the stable map compactification is greater than the dimension of the moduli space. This problem already appeared for the moduli space relative a smooth divisor, where the solution was to rescale the target normal to V to prevent components from sinking into V ; but now the problem is further compounded by the presence of the singular locus of V . So the main issue now is to how to precisely refine the Gromov compactness and construct an appropriate relatively stable map compactification $\overline{\mathcal{M}}_s(X, V)$ in such a way that its boundary strata are not larger dimensional than the interior.

In his unpublished Ph. D. thesis, Joshua Davis [Da] described how one can construct a relatively stable map compactification for the space of genus zero maps relative a normal crossings divisor, by recursively blowing up the singular locus of the divisor. As components sunk into this singular locus, he recursively blew it up to prevent this from happening. This works for genus zero, but unfortunately not in higher genus. The main reason for this is that in genus zero a dimension count shows that components sinking into V cause no problem, only those sinking into the singular locus of V do. However, that is not the case in higher genus, so then one would also need to rescale around V to prevent this type of behavior. But then the process never terminates: Josh had a simple example in higher genus where a component would sink into the singular locus. Blowing up the singular locus forced the component to fall into the exceptional divisor. Rescaling around the exceptional divisor then forced the component to fall back into the next singular locus, etc.

In this paper we present a different way to construct the a relatively stable map compactification $\overline{\mathcal{M}}_s(X, V)$, by instead rescaling X simultaneously normal to all the branches of V , a procedure we describe in Section 3. When done carefully, this is essentially a souped up

version of the rescaling procedure described in [IP1] in the case V was smooth. Unfortunately, the naive compactification that one would get by simply importing the description of that in [IP1] simply does not work when the singular locus of V is nonempty! There are two main reasons for its failure: the first problem is that the "boundary stratum" containing curves with components over the singular locus is again larger dimensional than the "interior" so it is in some sense too big; the second problem is that it still does not capture all the limits of curves sinking into the singular locus, so it is too small! This seems to lead into a dead end, but upon further analysis in Sections 5 and 6 of the limiting process near the singular locus two new features appear that allows us to still proceed.

The first new feature is the enhanced matching condition that the limit curves must satisfy along the singular locus of V . It turns out that not all the curves which satisfy the naive matching conditions can appear as limits of maps in $\mathcal{M}_s(X, V)$. The naive matching conditions require that the curves intersect V in the same points with matching order of contact, as was the case in [IP1], while the enhanced ones along the singular locus require in some sense that their slopes in the normal directions to V also match. So the enhanced matching conditions also involve the leading coefficients of the maps in these normal directions, and so they give conditions in a certain weighted projectivization of the normal bundle to the singular locus, a simple form of which is described in Section 4. Luckily, this is enough to cut back down the dimensions of the boundary to what should be expected. In retrospect, these enhanced matching conditions already appeared in one of the key Lemmas in our second joint paper [IP2] with Thomas H. Parker about the symplectic sum formula, but they do not play any role in the first paper [IP1] because they are automatically satisfied when V is smooth.

The second new feature that appears when V is singular is that unfortunately one cannot avoid trivial components stuck in the neck (over the singular locus of V), as we show in some simple Examples at the end of Section 4. This makes the enhanced matching conditions much more tricky to state, essentially because these trivial components do not have the right type of leading coefficients. The solution to this problem is to realize that the trivial components are there only to make the maps converge in Hausdorff distance to their limit, and in fact they do not play any essential role in the compactification, so one can simply collapse them in the domain, at the expense of allowing a node of the collapsed domain to be between not necessarily consecutive levels. The enhanced matching condition then occurs only at nodes between two nontrivial components, but needs to take into account this possible jump across levels. It is described more precisely in Section 5.

This finally allows us to define in Section 6 the compactified moduli space $\overline{\mathcal{M}}_s(X, V)$ of relatively V -stable maps into X , which comes together with a continuous map

$$\text{st} \times \text{Ev} : \overline{\mathcal{M}}_s(X, V) \rightarrow \overline{\mathcal{M}}_{\chi_s, \ell(s)} \times \prod_x \mathbb{P}_{s(x)}(NV_{I(x)}) \quad (0.1)$$

The first factor is the usual stabilization map recording the domain of f , which may be disconnected, but the new feature is the second factor Ev . It is a refinement of the usual (naive) evaluation map ev at the points x that are mapped into the singular locus of V , and it also records the weighted projectivization of the leading coefficients of f at x in all the normal directions to V at $f(x)$. This is precisely the map that appears in the enhanced matching conditions.

In Section 7 we then show that for generic V -compatible (J, ν) the image of $\overline{\mathcal{M}}_s(X, V)$ under the map (0.1) indeed defines a homology class $GW_s(X, V)$ in dimension

$$\dim \overline{\mathcal{M}}_s(X, V) = 2c_1(TX)A_s + (\dim X - 6)\frac{\chi_s}{2} + 2\ell(s) - 2A_s \cdot V$$

called the GW invariant of X relative the normal crossings divisor V . The class $GW_s(X, V)$ is independent of the perturbation ν and is in fact an invariant under smooth deformations of the pair (X, V) and of (ω, J) though V -compatible structures. When V is smooth these invariants agree with the usual relative GW invariants as constructed in [IP1].

There is a string of very recent preprints that have some overlap with the situation considered in our paper, in that they all generalize in some way the normal crossings situation from algebraic geometry. First of all, there is certainly an overlap between what we call a symplectic normal crossings divisor in this paper and what fits into the exploded manifold setup considered by Brett Parker [P]. There is also some overlap with the logarithmic Gromov-Witten invariants [GrS] considered by Gross and Siebert in the context of algebraic geometry (see also the Abramovich-Chen paper [AC] on a related topic). However, the precise local structure near the divisor is very different: log geometry vs symplectic normal crossings vs exploded structures. Furthermore, the moduli spaces constructed in these papers and in particular their compactifications are completely different, even when applied to the common case when V is a smooth divisor in a smooth projective variety, see Remarks 1.15 and 1.16 for more details. This means that a priori even in this common case each one of these other approaches many lead to different invariants, some even different from the usual relative GW invariants.

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1 Symplectic normal crossings divisors

In this section we define a notion of symplectic normal crossings divisors, which encodes the geometrical information required for the analysis of [IP1] and [IP2] to extend after appropriate modifications. In particular this notion generalizes the notion of normal crossings divisor in algebraic geometry. Clearly the local model of such divisor V should be the union of k coordinate planes in \mathbb{C}^n , where the number of planes may vary from point to point. But we also need a local model for the symplectic form ω and the tamed almost complex structure J near such divisor. We will therefore require that each branch of V is both ω symplectic and J holomorphic. This will allow us to define the order of contact of J -holomorphic curves to V . We also need a good description of the normal directions to the divisor, because these are going to be the directions in which the manifold X will be rescaled when components of the holomorphic curves fall into V . In particular, we need to keep track of both the normal bundle to each branch of V and its inclusion into X which describes the neighborhood of that branch.

Definition 1.1 (*local model*). *In \mathbb{C}^n consider the union V of $k \geq 0$ (distinct) coordinate hyperplanes $H_i = \{x|x_i = 0\}$ in \mathbb{C}^n , together with their normal direction N_i given by the usual projection $\pi_i : \mathbb{C}^n \rightarrow H_i$ onto the coordinate plane and the usual inclusion $\iota : N_i \rightarrow \mathbb{C}^n$. We say that they form a model for a normal crossings divisor in \mathbb{C}^n with respect to a pair (ω, J) if all the divisors H_i are both ω -symplectic and J -holomorphic.*

Remark 1.2 There is a natural action of \mathbb{C}^* on the model induced by scaling by a factor of t^{-1} in the normal direction to each H_i , for $i = 1, \dots, k$. This defines a rescaling map

$R_t : \mathbb{C}^n \rightarrow \mathbb{C}^n$ for $t \in \mathbb{C}^*$. By construction, the R_t leaves the divisors H_i invariant, but not pointwise, and may not preserve J . However, as $t \rightarrow 0$, $R_t^* J$ converges uniformly on compacts to a \mathbb{C}^* invariant limit J_0 which depends on the 1-jet of J along the divisor.

Definition 1.3 Assume (X, ω, J) is a symplectic manifold with a tamed almost complex structure. V is called a normal crossings divisor in (X, ω, J) with normal bundle N if there exists a smooth manifold \tilde{V} with a complex line bundle $\pi : N \rightarrow \tilde{V}$ and an immersion $\iota : U_V \rightarrow X$ of some disk bundle U_V of N_V into X satisfying the following properties:

- V is the image of the zero section \tilde{V} of N
- the restriction of $\iota^* J$ to the fiber of N along the zero section induces the complex multiplication in the bundle N .
- at each point $p \in X$ we can find local coordinates on X in which the configuration (X, π, ι, V) becomes identified with one of the local models in Definition 1.1.

Such a pair (J, ω) is called adapted to the divisor V .

Note that ι induces by pullback from X both a symplectic structure ω and an almost complex structure J on the total space of the disk bundle in N over on \tilde{V} , which will serve as a global model of X near V . Its zero section \tilde{V} is both symplectic and J -holomorphic and serves as a smooth model of the divisor V (called the *normalization* of V). N is also a complex line bundle whose complex structure comes from the restriction of J along the zero section. Thus N also comes with a \mathbb{C}^* action which will be used to rescale X normal to V .

Remark 1.4 We are not requiring J to be locally invariant under this \mathbb{C}^* action. We also are not imposing the condition that the branches are perpendicular with respect to ω or that the projections π_i are J -holomorphic. We also allow for self intersections of various components of V . When each component of V is a submanifold of X the divisor is said to have *simple* normal crossings singularities. Any of these assumptions would simplify some of the arguments, but are not needed.

In this paper we will work only with J 's which are compatible to V in the sense of Definition 3.2 of [IP1]. This is a condition on the normal 1-jet of J along V :

$$(b) \quad [(\nabla_\xi J + J \nabla_{J\xi} J)(v)]^N = [\nabla_v J] \xi + J(\nabla_{Jv} J) \xi]^N \text{ for all } v \in TV, \xi \in NV;$$

discussed in more detail in the Appendix. This extra condition is needed to ensure that the stable map compactification has codimension 2 boundary strata, so it gives an invariant, independent of parameters. A priori, even when V is smooth the relatively stable map compactification may have real codimension 1 boundary without this extra assumption.

Example 1.5 A large class of examples of normal crossings is provided by algebraic geometry. Assume X is a smooth projective variety and V a complex codimension 1 subvariety which is locally a transverse union of smooth divisors. In particular V could be the transverse intersection of smooth divisors, in which case V is said to have *simple* normal crossings, but in general the divisors may also self intersect. Then V is a symplectic normal crossings divisor for (X, J_0, ω_0) where J_0 is integrable complex structure and ω_0 the Kahler form. For example (a) X could be a Hirzebruch surface and V the union of the zero section, the infinity section and several fibers or (b) V could be the union of a section and a nodal fiber in an elliptic surface X .

An important example of this type is when X is a toric manifold and V is its toric divisor, which is a case considered in mirror symmetry, see for example [Au2].

Example 1.6 Another particular example to keep in mind is $X = \mathbb{CP}^2$ with a degree 3 normal crossings divisor V . For example V could be a smooth elliptic curve, or V could be a nodal sphere, or finally V could be a union of 3 distinct lines. In the second case the normalization \tilde{V} is \mathbb{CP}^1 with normal bundle $O(7)$ while in the last case it is $\mathbb{CP}^1 \sqcup \mathbb{CP}^1 \sqcup \mathbb{CP}^1$, each component with normal bundle $O(1)$. Of course, in a complex 1-parameter family, a smooth degree three curve can degenerate into either one of the other two cases.

Another motivating example of this type comes from a smooth quintic 3-fold degenerating to a union of 5 hyperplanes in \mathbb{CP}^4 .

Remark 1.7 Yet another special case is $X = \overline{\mathcal{M}}_{0,n}$ the Deligne-Mumford moduli space of stable genus 0 curves and V the union of all its boundary strata (i.e. the stratum of nodal curves). The usual description of each boundary stratum and of its normal bundle provides the required local models for a symplectic normal crossings divisor. This discussion can also be extended to the orbifold setting to cover the higher genus case $\overline{\mathcal{M}}_{g,n}$ and certainly covers its smooth finite cover, the moduli space of Prym curves, constructed by Looijenga [Lo].

Of course, there are many more symplectic examples besides those coming from algebraic geometry.

Example 1.8 Assume V is a symplectic codimension two submanifold of (X, ω) . The symplectic neighborhood theorem then allows us to find a J and a model for the normal direction to V , so V is normal crossings divisor in (X, ω, J) . Of course in this case V is a smooth divisor, so it has empty singular locus.

One may have hoped that the union of several transversely intersecting codimension two symplectic submanifolds would also similarly be a normal crossings divisor. Unfortunately, if the singular locus is not empty, that may not be the case, as illustrated by the example below.

Example 1.9 Let V_1 be an exceptional divisor in a symplectic 4-manifold and V_2 a sufficiently small generic perturbation of it, thus still a symplectic submanifold, intersecting transversely V_1 . This configuration cannot be given the structure of a normal crossings divisor, simply because one cannot find a J which preserves both. If such a J existed, then all the intersections between V_1 and V_2 would be positive, contradicting the fact that exceptional divisors have negative self intersection.

This example illustrates the fact that a normal crossings divisor is not a purely symplectic notion, but rather one also needs the existence of a tamed almost complex structure J adapted to the crossings. The positivity of intersections of all branches is a necessary condition for such a J to exist in general.

Remark 1.10 One could ask what are the necessary and sufficient conditions for V inside a symplectic manifold (X, ω) to be a normal crossings divisor with respect to some J on X . Clearly V should be locally the transverse intersection of symplectic submanifolds, and furthermore this intersection should be positive. If we assume that the branches of V are moreover orthogonal wrt ω then the existence of a compatible J is straightforward (see the end of Appendix).

Example 1.11 Symplectic Lefschetz pencils or fibrations provide another source of examples of symplectic normal crossings divisors. Assume X is a symplectic manifold which has a symplectic Lefschetz fibration with a symplectic section, for example one coming from Donaldson Theorem [Do2] where the section comes from blowing up the base locus. Gompf [GoS] showed that in this case there is an adapted almost complex structure J to this fibration. We

could then take V the union of the section with a bunch of fibers, including possibly some singular fibers.

Example 1.12 (Donaldson divisors) Assume V is a normal crossings divisor in (X, ω, J) and that $[\omega]$ has rational coefficients. We can use Donaldson theorem [Do] to obtain a smooth divisor D representing the Poincare dual of $k\omega$ for $k \gg 0$ sufficiently large, such that D is ε - J -holomorphic and η -transverse to V (see also [Au]). Choosing carefully the parameters η and ε , one can then find a sufficiently small deformation of J such that $V \cup D$ is also a normal crossings divisor.

Remark 1.13 The definition of a normal crossings divisor works well under taking products of symplectic manifolds with divisors in them. If V_i is a normal crossings divisor in X_i for $i = 1, 2$ then $\pi_1^{-1}(V_1) \cup \pi_2^{-1}(V_2) = V_1 \times X_2 \cup X_1 \times V_2$ is a normal crossings divisor in $X_1 \times X_2$, with normal model $\pi_1^*N_1 \sqcup \pi_2^*N_2$. Note that even if V_i were smooth divisors, then the divisor in the product $X_1 \times X_2$ is still singular along $V_1 \times V_2$.

Remark 1.14 The definition of a normal crossings divisor also behaves well under symplectic sums. Assume $U_i \cup V$ is a symplectic divisor in X_i for $i = 1, 2$ such that the normal bundles of V in X_i are dual. If U_i intersect V in the same divisor W then Gompf's argument [Go] shows that the divisors U_i glue to give a normal crossings divisor $U_1 \#_W U_2$ in the symplectic sum $X_1 \#_V X_2$.

Remark 1.15 A special case of symplectic normal crossings divisor V (with simple crossings) is the union of codimension 2 symplectic submanifolds which intersect orthogonally wrt ω , and whose local model matches that of toric divisors in a toric manifold. This is a case that fits in the exploded manifold set-up of Brett Parker (see Example 5.3 in the recent preprint [P]), so in principle one should be able to compare the relative invariants we construct in this paper with the exploded ones of [P]. It is unclear to us what is exactly the information that the exploded structure records in this case, and what is the precise relation between the two moduli spaces. But certainly the relatively stable map compactification we define in this paper seems to be quite different from the exploded one, so it is unclear whether they give the same invariants, even in the case when V is smooth.

Remark 1.16 In a related recent preprint, Gross and Siebert define log GW invariants in the algebraic geometry setting [GrS]. If V is a normal crossings divisor in a smooth projective variety X , then it induces a log structure on X . However, even in the case V is a smooth divisor, Gross and Siebert explain that the stable log compactification they construct is quite different from the relatively stable map compactification constructed earlier in that context by J. Li [Li] (and which agrees with that of [IP1] in this context). So a priori, even when V is smooth, the usual relative GW invariants may be different from the log GW invariants of [GrS]. The authors mention however that in that case at least there is a map from the moduli space of stable relative maps to that of stable log maps, which they claim could be used to prove that the invariants are the same, though no proof of this claim is available yet. Presumably there is also a map from the relatively stable map compactification that we construct in this paper to the appropriate stable log compactification in the more general case when V is a normal crossings divisor in a smooth projective variety.

In a related paper [AC] Abramovich and Chen also explain how, in the context of algebraic geometry, the construction of a log moduli space when V is a normal crossings divisor (with simple crossings) follows from the case when V is smooth by essentially functorial reasons. Again, it is unclear to us how exactly the two notions of log stable maps of [GrS] and [AC] are related in this case.

Remark 1.17 One note about simple normal crossings vs general normal crossings: they do complicate the topology/combinatorics of the situation, but if set up carefully the analysis is unaffected. If the local model of X is holomorphic near V (as is the case in last two examples above), even if V did not have simple crossings, one could always blow up the singular locus W of V to get a total divisor $\pi^{-1}(V) = Bl(V) \cup E$ with simple normal crossings in $Bl(X)$, where E is the exceptional divisor. Blowing up in the symplectic category is a more delicate issue, but when using the appropriate local model, one can always express (a symplectic deformation) of the original manifold (X, V) as a symplectic sum of its blow up $(Bl(X), Bl(V))$ along the exceptional divisor E with a standard piece (\mathbb{P}, V_0) involving the normal bundle of the blowup locus. Since we are blowing up the singular locus of V , the proper transform $Bl(V)$ intersects nontrivially the exceptional divisor E ; the symplectic sum $Bl(X) \#_E \mathbb{P} = X$ then also glues $Bl(V)$ to the standard piece on the other side to produce V , as in Remark 1.14. Therefore a posteriori, after proving a symplectic sum formula for the relative GW of normal crossings divisors passing through the neck of a symplectic sum, one could also express the relative GW invariants of the original pair (X, V) as universal expressions in the relative GW invariants of its the blow up and those of the piece obtained from the normal bundle of the blow-up locus.

We study some of the properties of (symplectic) normal crossings divisors in more detail in the Appendix. This is also where we include a more detailed description of the stratifications of the divisor V which record how the various local branches of V intersect. In particular, each stratum is itself a normal crossings divisor in the resolution of the next one, and we describe its normalization. These provide a global way to encode the local information about the way J -holomorphic curves meet various branches of V .

2 The Relative Moduli Space $\mathcal{M}_s(X, V)$

Assume now (X, ω, J) is a smooth symplectic manifold with a normal crossings divisor V . We want to define the moduli space of stable (perturbed) J -holomorphic maps into X relative V , which will enter in the definition of the relative GW invariant of the pair (X, V) in the usual way. We will follow closely the approach of [IP1] and [IP2] and explain how many of the arguments there extend in this context.

The notion of stability always refers to finitely many automorphisms, but the notion of automorphism is relative, that is it depends on the particular setup. We will later on explain what exactly we mean by a stable map $f : C \rightarrow X$ relative V ; but any automorphism of the map will in particular be an automorphism of its domain C .

Remark 2.1 (Stability and automorphisms) For the purpose of simplifying the discussion in this paper, in all the local analysis arguments below we will implicitly assume that all the domains C are already stable, and have been furthermore decorated to have non nontrivial automorphisms, as is discussed in Section 1 of [IP1]. First of all, any unstable domain components of C are spheres, and collapsing them gives an element $\text{st}(C)$ of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. If we assume to begin with that we are in a situation where all domains are stable, then after possibly decorating the domains by adding a finite amount of extra topological information (like a Prym structure) we can also assume they now have no nontrivial automorphisms; in particular, their moduli space $\overline{\mathcal{M}}$ is smooth. In genus zero it is already the case that any stable domain has no nontrivial automorphisms, and the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,n}$ is smooth; in higher genus we can replace $\overline{\mathcal{M}}_{g,n}$ by its smooth finite cover, the moduli space of Prym curves constructed by Looijenga in [Lo]. This will have the effect of globally killing all nontrivial automorphisms of stable domains,

thus making analysis arguments like transversality much easier to setup, at the expense of passing to a finite branched cover of the original moduli space.

Fix also a particular embedding

$$\overline{\mathcal{U}} \rightarrow \mathbb{P}^N \quad (2.1)$$

of the universal curve $\overline{\mathcal{U}}$ over the Deligne-Mumford moduli space in genus zero or over the Prym moduli space in higher genus. The embedding (2.1) gives a canonical choice of a complex structure j on each domain C obtained by restricting the complex structure on \mathbb{P}^N to the fiber $st(C)$ of the universal curve; the unstable domain components, if any, already have a canonical j_0 on them. So the embedding (2.1) provides a *global slice* to the action of the reparametrization group on the moduli space of maps, as long as their domains are stable; otherwise there is still a residual action coming from automorphisms of the unstable components. This embedding also simultaneously gives us a simple type of global perturbation ν of the holomorphic map equation

$$\overline{\partial}_{j,J} f(z) = \nu(z, f(z)) \quad (2.2)$$

coming from $\overline{\mathcal{U}} \times X$ as it sits as a smooth submanifold inside $\mathbb{P}^N \times X$. The perturbation ν then vanishes on all the unstable components of the domain, so these are J -holomorphic. In the case all the domains are stable to begin with, or more generally when the restriction of f to the unstable part of the domain is a simple J -holomorphic map, this type of perturbation ν is enough to achieve all the required transversality; in general there is still a problem achieving transversality using this type of perturbation on the unstable part of the domain that is multiply covered.

There are many ways to locally stabilize the domains to locally get oneself in the situation described above; understanding how these local choices globally patch together is at the heart of the construction of the virtual fundamental cycle in GW theories, see Remark 1.9 of [IP1] and the references therein. For the situation we discuss in this paper, we could for example add a Donaldson divisor to V as in Example 1.12; this will have the effect that all the maps in the relatively stable map compactification will now have stable domains. It is not a priori clear why one would get an invariant that way (independent of the Donaldson divisor added), but that will be the topic of another upcoming paper.

Furthermore, in this paper we will work only with V -compatible parameters (J, ν) for the equation (2.2) as in [IP1]. So we will assume that (J, ν) satisfy the conditions of Definition 3.2 in [IP1] the normal direction to each branch of V . These are conditions only on the 1-jet of (J, ν) along V , and are used to show that the contact order to V is well defined and that generically the stable map compactification does not have real codimension one boundary strata. See Remark A.3. in the Appendix for a discussion of the space of V -compatible parameters (J, ν) .

2.1 The relative moduli space and the contact information

The construction of the relative moduli space takes several stages; in this section we describe the main piece

$$\mathcal{M}_s(X, V)$$

containing stable (J, ν) -holomorphic maps $f : C \rightarrow X$ into X without any components or nodes in V , such that all the points in $f^{-1}(V)$ are marked, and come decorated by the sequence s of multiplicities recording the order of contact of f to each local branch of V .

There is a fair amount of discrete topological/combinatorial data that needs to be kept track of in the definition of \mathcal{M}_s . We also discuss the construction of the leading order section which keeps track of more information than the ordinary evaluation map, and construct an enhanced evaluation map using it. This information will then be crucial in the later sections when we construct the relatively stable map compactification.

We start by explaining how the discussion in Section 4 of [IP1] extends to the case when V is a normal crossings divisor. Assume $f : C \rightarrow X$ is a stable (J, ν) holomorphic map into X that has no components or nodes in V , such that **all** the points in $f^{-1}(V)$ are marked. Assume x is one of the marked points such that $f(x)$ belongs to the (open) stratum of V where k local branches meet.

Choose local coordinates z about the point x in the domain, and locally index the k branches of V meeting at $f(x)$ by some set I . See the discussion in the Appendix on how to regard the indexing set I in an intrinsic manner. For each $i \in I$, choose also a local coordinate at $f(x)$ in the normal bundle to the branch labeled by i , see also (A.4.). Lemma 3.4 of [IP1] then implies that the normal component f_i of the map f around $z = 0$ has an expansion:

$$f_i(z) = a_i z^{s_i} + O(|z|^{s_i}) \quad (2.3)$$

where s_i is a positive integer and the leading coefficient $a_i \neq 0$. The multiplicity s_i is independent of the local coordinates used, and records the order of contact of f at x to the i 's local branch of V .

Thus each point $x \in f^{-1}(V)$ comes with the following information:

- (a) a depth $k(x) \geq 1$ that records the codimension in X of the open stratum of V containing $f(x)$;
- (b) an indexing set $I(x)$ of length $k(x)$ that keeps track of the local branches of V meeting at $f(x)$;
- (c) a sequence of positive multiplicities $s(x) = (s_i(x))$ indexed by $i \in I(x)$, recording the order of contact of f at x to each of the $k(x)$ local branches of V .

We will think of the indexing set $I(x)$ as part of the information contained in the sequence $s(x)$. There are several ways to encode this: one way is regard $s(x)$ as a map $s(x) : I(x) \rightarrow \mathbb{N}_+^{k(x)}$ defined by $i \mapsto s_i(x)$, the multiplicity of contact at x to the branch indexed by $i \in I(x)$. Either way, the sequence $s(x)$ keeps track of how the local intersection number of f at x with V is partitioned into intersection numbers to each local branch of V . The sequence s is obtained by putting together all the $s(x)$ for all $x \in f^{-1}(V)$ and keeps track of all the contact information of f to V .

Remark 2.2 To keep notation manageable we will also include in s the information about ordinary marked points (i.e. for which $f(x) \notin V$). For such points, we make the convention that the depth $k = 0$, $I = \emptyset$ and $s(x) = \emptyset$. Then each marked point x now has a depth $k \geq 0$: depth 0 corresponds to an ordinary marked point, mapped into $X \setminus V$, depth 1 is mapped into the smooth locus of V and has attached just one positive multiplicity, etc.

This concludes our description of the moduli space

$$\mathcal{M}_s(X, V)$$

consisting of stable (J, ν) -holomorphic maps $f : C \rightarrow X$ that have no components or nodes in V and such that all the points in the inverse image of V are marked points of C and

furthermore decorated according to the sequence s . If P denotes the collection of marked points of C , each marked point $x \in P$ has an associated sequence $s(x)$ which records the order of contact of the map f at x to each local branch of V , including the information about the indexing set $I(x)$ of the branches. The cardinality $k(x) = |I(x)|$ of $I(x)$ records the **depth** of x while the **degree** of $s(x)$

$$|s(x)| \stackrel{\text{def}}{=} \sum_{i \in I(x)} s_i(x)$$

is the local intersection number of f at x with V . In particular, the total degree of s is purely topological:

$$|s| = \sum_{x \in P} |s(x)| = f^*(V) = A \cdot V \quad (2.4)$$

where $A \in H_2(X)$ is the homology class of the image of f . We also denote by $\ell(s)$ the **length** of s , i.e. the total number of marked points $x \in P$ of C .

Remark 2.3 To keep the notation simple, we will think of s as recording ALL the topological information about the stable maps $f : C \rightarrow X$; in particular this includes the *homology class* A_s of the image of f and the topological type Σ_s of the domain of f , and so its *Euler characteristic* $\chi_s = \chi(\Sigma_s)$. In the discussion above, the domain Σ_s of f could be disconnected, in which case its components are unordered. We can also include in s not just the homology class of the image of f , but also its appropriate relative homology class, i.e. the information about the rim tori, see Section 5 of [IP1]. The construction there is purely topological, so it easily extends to the case when V is a normal crossings divisor. We will not explicitly describe it in this paper.

With this notation, the arguments of Lemma 4.2 in [IP1] immediately extend in this context to show that for generic V -compatible (J, ν) the moduli space $\mathcal{M}_s(X, V)$ is a smooth orbifold of dimension

$$\dim \mathcal{M}_s(X, V) = 2c_1(TX)A_s + (\dim X - 6)\frac{\chi(\Sigma_s)}{2} + 2\ell(s) - 2A_s \cdot V \quad (2.5)$$

Note that this dimension does not depend on the particular partition of the intersection number of V and A , only on the total number $\ell(s)$ of marked points (which include ordinary marked points in our convention).

The *ordinary evaluation map* at one of the decorated marked points $x \in P$ is defined by

$$\begin{aligned} \text{ev}_x : \mathcal{M}_s(X, V) &\rightarrow V_{I(x)} \\ \text{ev}_x(f) &= f(x) \end{aligned} \quad (2.6)$$

where $V_{I(x)}$ is the depth $k(x)$ stratum of V labeled by the indexing set $I(x)$ of the branches of V . As before, this evaluation map includes the ordinary marked points when we make the convention that $V_\emptyset = X$. Putting together all the marked points $x \in P(s)$ labeled by the sequence s we get the corresponding ordinary evaluation map

$$\text{ev} : \mathcal{M}_s(X, V) \rightarrow V_s = \prod_{x \in P(s)} V_{I(x)} \quad (2.7)$$

whose image is the product of all the strata corresponding to the marked points. It is important to note the image of the evaluation map above: there are several other choices that may seem possible (like simply $X^{\ell(s)}$), but this is the only choice for which the evaluation map can be transverse, without losing important information.

Remark 2.4 In the above discussion for simplicity we identified the marked point x which is an actual point in the domain of f with its marking (index) in P ; usually one talks about the p 'th marked point x_p and about ev_p , the evaluation at the p 'th marked point.

When the depth $k(x) \geq 2$, the evaluation map (2.7) still does not record enough information to state the full matching conditions that appear in the relatively stable map compactification. We will also need to record the leading coefficient of the expansion (2.3). For each $f : C \rightarrow X$ in $\mathcal{M}_s(X, V)$ and each marked point $x \in f^{-1}(V)$ let $a_i(x)$ be leading coefficient (2.3) of f at x in the normal direction N_i to the branch labeled by $i \in I(x)$. As explained in Section 5 of [IP2], $a_i(x)$ is naturally an element of $(N_i)_{f(x)} \otimes (T_x^* C)^{s_i(x)}$ so it defines a section of the bundle

$$\text{ev}_x^* N_i \otimes L_x^{s_i(x)} \quad (2.8)$$

where L_x is the relative cotangent bundle to the domain at the marked point x . If we denote by $E_{s,x}$ the bundle over the moduli space whose fiber at f is

$$\bigoplus_{i \in I(x)} \text{ev}_x^* N_i \otimes L_x^{s_i(x)} \quad (2.9)$$

then the *leading order section* at x is defined by

$$\begin{aligned} \sigma_x : \mathcal{M}_s(X, V) &\rightarrow E_{s,x} \\ \sigma_x(f) &= (a_i(x))_{i \in I(x)} \end{aligned} \quad (2.10)$$

and records the leading term coefficient in each one of the $k(x)$ normal directions at x labeled by $I(x)$. This section will turn out to record a lot of crucial information, and will be studied in more detailed later. It was already used in [I] to essentially get an isomorphism between the relative cotangent bundle of the domain and that of the target for the moduli space of branched covers of \mathbb{P}^1 .

Remark 2.5 The target of (2.10) may not be globally a direct sum of line bundles, as its fibers may intertwine as we move along $V_{I(x)}$, the same way the $k(x)$ branches of V do when the indexing set $I(x)$ has nontrivial global monodromy, see discussion in the Appendix.

The *enhanced evaluation map* is defined by

$$\begin{aligned} \text{Ev}_x : \mathcal{M}_s(X, V) &\rightarrow \mathbb{P}_{s(x)}(NV_{I(x)}) \\ \text{Ev}_x(f) &= [\sigma_x(f)] \end{aligned} \quad (2.11)$$

where the $\mathbb{P}_{s(x)}(NV_{I(x)})$ denotes the weighted projectivization with weight $s(x)$ of the normal bundle $NV_{I(x)}$ of the depth $k(x)$ stratum $V_{I(x)}$. More precisely, $\mathbb{P}_{s(x)}(NV_{I(x)})$ is a bundle over $V_{I(x)}$ whose fiber is the weighted projective space obtained as the quotient by the \mathbb{C}^* action with weight $s_i(x)$ in the normal direction N_i to the branch labeled by $i \in I(x)$. Globally these branches may intertwine as discussed above in Remark 2.5.

Of course, if $\pi : \mathbb{P}_{s(x)}(NV_{I(x)}) \rightarrow V_{I(x)}$ is the projection then

$$\pi \circ \text{Ev} = \text{ev}$$

which explains the name; Ev_x is a refinement of ev_x only when the depth $k(x) \geq 2$.

Remark 2.6 Note that by construction the leading order terms are all nonzero, so the image of σ_x is away from the zero sections of each term in (2.10). The image of the enhanced evaluation map similarly lands in the complement of all the coordinate hyperplanes in the target of (2.11).

So far we defined the moduli space $\mathcal{M}_s(X, V)$ of stable maps into X without any components or nodes in V , and with fixed topological information described by s . This moduli space $\mathcal{M}_s(X, V)$ may not be compact, simply because we could have sequence of maps f_n in \mathcal{M}_s whose limit in the stable maps compactification is a map f with some components in V . Then the contact information of f to V becomes undefined along its components that lie in V , and so in particular f does not belong to $\mathcal{M}_s(X, V)$. Note that this is the only reason why $\mathcal{M}_s(X, V)$ fails to be compact:

Lemma 2.7 *Consider a sequence $\{f_n\}$ of maps in $\mathcal{M}_s(X, V)$ and assume its limit f in the usual stable map compactification has no components in V . Then $f \in \mathcal{M}_s(X, V)$.*

Proof. A priori, there are two reasons why f would fail to be in \mathcal{M}_s :

- (a) f has a node in V or
- (b) the contact information of f is not given by s .

Note that case (b) includes the cases when in the limit multiplicity of intersection jumps up or when a depth k marked point falls into a higher depth stratum of V .

Since ALL points in $f_n^{-1}(V)$ are already marked, indexed by the same set P , they persist as marked points for the limit f , in particular they are distinct from each other and from the nodes of f . On the other hand, let \tilde{f} denote the lift of f to the normalization of its domain. Since f has no components in V , then each point in $\tilde{f}^{-1}(V)$ has a well defined sequence s_0 that records the local multiplicity of intersection of \tilde{f} at that point with each local branch of V . At those points of $\tilde{f}^{-1}(V)$ which were limits of the marked points in $f_n^{-1}(V)$, the multiplicity $s_0(x) \geq s(x)$, as the mutiplicity could go up when the leading coefficients converge to 0. But then

$$[f] \cdot V = \sum_{x \in \tilde{f}^{-1}(V)} s_0(x) \geq \sum_{x \in P} s_0(x) \geq \sum_{x \in P} s(x) = [f_n] \cdot V$$

Since $[f_n] = [f]$ then $\tilde{f}^{-1}(V) = P$, which means that f has no nodes in V , rulling out (a), and that $s_0(x) = s(x)$ for all $x \in P$, which rules out (b). \square

3 Rescaling the target

Assume now $\{f_n\}$ is a sequence of maps in $\mathcal{M}_s(X, V)$ whose limit f in the stable maps compactification has some components in V . We will use the methods of [IP1], to rescale the target normal to V to prevent this from happening. The analysis there is mostly done semi-locally (in a neighborhood of V) and so those parts easily extend to this situation. But as we will see below, the topology of the normal crossings divisor now enters crucially in a couple of steps. First step is to describe the effect of rescaling on the target X . It is modeled on the process of rescaling a disk about the origin, but now performed fiberwise in the normal direction to V .

3.1 Brief review of [IP1]

We begin by briefly reviewing the situation in Section 6 of [IP1], where it was assumed that V is a smooth divisor. In local coordinates, if x is a fixed local coordinate normal to V ,

rescaling X by a factor of $\lambda \neq 0$ means we make a change of coordinates in a neighborhood of V in the normal direction to V :

$$x_\lambda = x/\lambda. \quad (3.1)$$

Under rescaling by an appropriate amount λ_n , depending on the sequence $\{f_n\}$, in the limit we will get not just a curve in X (equal to the part of f that did not lie in V), but also a curve in the compactification of N_V , i.e. in

$$\mathbb{F} = \mathbb{P}(N_V \oplus \mathbb{C}).$$

Here \mathbb{F} is a \mathbb{CP}^1 bundle over V , with a zero and infinity section V_0 and V_∞ . Under the rescaling (3.1), x_λ can be thought instead as a coordinate on \mathbb{F} normal to V_0 . Let $y = 1/x_\lambda$ be the corresponding coordinate normal to V_∞ inside \mathbb{F} , so that (3.1) becomes

$$xy = \lambda \quad (3.2)$$

This procedure has the infinity section V_∞ of \mathbb{F} naturally identified with V in X such that furthermore their normal bundles are dual to each other, i.e.

$$N_{V/X} \otimes N_{V_\infty/\mathbb{F}} \cong \mathbb{C} \quad (3.3)$$

is trivial. This identification globally encodes the local equations (3.2) because x, y are local sections of $N_{V/X}$ and $N_{V_\infty/\mathbb{F}}$ respectively.

Remark 3.1 Once an identification (3.3) is fixed, then for any (small) gluing parameter $\lambda \in \mathbb{C}^*$, equation (3.2) is exactly the local model of the symplectic sum X_λ of X and \mathbb{F} along $V = V_\infty$ with gluing parameter λ . Of course, topologically $X \#_{V=V_\infty} \mathbb{F} = X$. This means that an equivalent point of view to rescaling X by a factor of λ normal to V is to regard X as the symplectic sum X_λ of X and \mathbb{F} with gluing parameter λ , with the above choice of coordinates and identifications, including (3.3). The advantage of this perspective is that the rescaled manifolds X_λ now fit together as part of a smooth total space Z as its fibers over λ , and converge there as $\lambda \rightarrow 0$ to the singular space

$$X_0 = X \bigcup_{V=V_\infty} \mathbb{F}, \quad (3.4)$$

obtained by joining X to \mathbb{F} along $V = V_\infty$.

Denote by U_λ the tubular neighborhood of V in X described in coordinates by $|x| \leq |\lambda|^{1/2}$ and by V_λ the complement of the tubular neighborhood of V_∞ in \mathbb{F} described in coordinates by $|y| \geq |\lambda|^{1/2}$. From this perspective, rescaling X around V by $\lambda \neq 0$ gives rise to a manifold X_λ together with a diffeomorphism

$$R_\lambda : X \rightarrow X_\lambda \quad (3.5)$$

which is the identity outside U_λ and which identifies U_λ with V_λ by rescaling it by a factor of λ , or equivalently via the equation (3.2). As $\lambda \rightarrow 0$, U_λ shrinks to V inside X , but it expands in the rescaled version X_λ to $\mathbb{F} \setminus F_\infty$. So in the limit as $\lambda \rightarrow 0$, the rescaled manifolds X_λ , with the induced almost complex structures $J_\lambda = (R_\lambda^{-1})^* J$ converge to the singular space X_0 defined by (3.4) with an almost complex structure J_0 which agrees with J on X and is \mathbb{C}^* invariant on the \mathbb{F} piece.

After rescaling the sequence $\{f_n\}$ by appropriate amounts λ_n , the new sequence $R_{\lambda_n}(f_n)$ has a limit inside Z which is now a map into X_0 satisfying a matching condition along $V_\infty = V$, described in more details later on. Of course, in general, different components may fall in at different rates, so we need to rescale several (but finitely many) times to catch all of them, and in the limit we get a map into a building with several levels.

3.2 Rescaling the manifold X normal to V

Assume now V is a normal crossings divisor. We next describe the effect on the manifold X of rescaling around V (in a tubular neighborhood of V). Using our local models, we could extend the discussion above independently in each normal direction to V , so normal to each open stratum of V^k , we could rescale in k independent directions. However, globally these directions may intertwine, and not be independent, so one has to be careful how to globally patch these local pictures. Here is where we use the fact that the normal bundle N was defined over the normalization \tilde{V} of V . We will rescale normal to V using the \mathbb{C}^* action in this normal bundle.

Remark 3.2 The \mathbb{C}^* action in the complex line bundle N induces in fact several different actions. The one we will use in this paper is the diagonal \mathbb{C}^* action in the normal bundle to each stratum V^k of V . When the normalization of V has several components we have a \mathbb{C}^* action for each component. In particular, when the divisor V has simple crossings, then we also have a local $(\mathbb{C}^*)^k$ action on X normal to each V^k ; essentially this happens only when the crossings are simple.

When we rescale once X normal to the normal crossings divisor V , in level one we get several pieces, one for each piece V^k of the stratification of V according to how many local branches meet there. The level zero unrescaled piece is still (X, V) as before. But now level one

$$\mathbb{F}_V = \bigsqcup_{k \geq 1} \mathbb{F}_k \quad (3.6)$$

consists of several pieces \mathbb{F}_k , one for each depth $k \geq 1$. The first piece is

$$\mathbb{F}_1 = \mathbb{P}(N_V \oplus \mathbb{C}),$$

a \mathbb{P}^1 bundle over the \tilde{V} , the normalization of V , obtained by compactifying the normal bundle $N_V \rightarrow \tilde{V}$ by adding an infinity section. Similarly, the k 'th piece

$$\mathbb{F}_k \longrightarrow \widetilde{V^k}$$

is a $(\mathbb{P}^1)^k$ bundle over the normalization $\widetilde{V^k}$ of the closed stratum V^k , described in more details in the Appendix. What is important here is that \mathbb{F}_k is a bundle over a smooth manifold $\widetilde{V^k}$ which it is obtained by separately compactifying each of the k normal directions to V along V^k , see (A.4.). This means that its fiber at a point $p \in \widetilde{V^k}$ is

$$\mathbb{F}_k = \times_{i \in I} \mathbb{P}(N_i \oplus \mathbb{C})$$

where N_i is the normal direction to the i 'th branch, and I is an indexing set of the k local branches of V meeting at p . Globally, these \mathbb{P}^1 factors intertwine as dictated by the global monodromy of the k local branches of V .

Each piece \mathbb{F}_k comes with a natural normal crossings divisor

$$W^k = D_{k,\infty} \cup D_{k,0} \cup F_k \quad (3.7)$$

obtained by considering together its zero and infinity divisors plus the fiber F_k over the (inverse image of the) higher depth strata of V^k . The construction of the divisor W^k is described in more details the Appendix, but here let us just mention that $D_{k,0}$ is the zero divisor in \mathbb{F}_k where at least one of the \mathbb{P}^1 coordinates is equal to 0, while the fiber divisor F_k is the restriction of \mathbb{F}_k to the stratum of $\widetilde{V^k}$ coming from the higher depth stratum V^{k+1} .

Remark 3.3 The \mathbb{C}^* action in the normal bundle to V induces a fiberwise, diagonal \mathbb{C}^* action on each piece \mathbb{F}_k ; the action preserves the divisor W^k , but not pointwise.

To keep notation manageable, we make the convention

$$(\mathbb{F}_0, W^0) \stackrel{\text{def}}{=} (X, V) \quad (3.8)$$

This is consistent with our previous conventions that $V^0 = X$, in which case $\mathbb{F}_0 = \mathbb{P}(N_{V^0/X} \oplus \mathbb{C}) = X$ and $W_0 = F_0 = V$ is the fiber divisor over the lower stratum, as the zero and infinity divisors are empty in this case. However, one difference that this notation obscures is the fact that while $X = \mathbb{F}_0$ is on level 0 (unrescaled), the rest of the pieces \mathbb{F}_k for $k \geq 1$ are all on level 1 (all appeared as the result of rescaling once normal to V).

Definition 3.4 A level one building X_1 with zero divisor V_1 is obtained by identifying the fiber divisor F_k of \mathbb{F}_k with the infinity divisor $D_{k+1,\infty}$ of \mathbb{F}_{k+1} :

$$X_1 = \bigcup_{F_k = D_{k+1,\infty}} \mathbb{F}_k$$

Denote by W_1 the singular locus of X_1 where all the pieces are attached to each other, by V_1 the (singular) divisor in X_1 obtained from the zero divisors, and let $D_1 = W_1 \cup V_1$ be the total divisor:

$$(X_1, D_1) = \bigcup_{F_k = D_{k+1,\infty}} (\mathbb{F}_k, F_k \cup D_{k,\infty} \cup D_{k,0}) \quad (3.9)$$

So the level one building comes with a resolution $(\tilde{X}_1, \tilde{D}_1) = \bigsqcup_{k \geq 0} (\mathbb{F}_k, F_k \cup D_{k,\infty} \cup D_{k,0})$ and an attaching map

$$\xi : (\tilde{X}_1, \tilde{V}_1) \rightarrow (X_1, V_1). \quad (3.10)$$

It also comes with a collapsing map to level zero $(X_0, V_0) = (X, V)$:

$$p : (X_1, V_1) \rightarrow (X, V) \quad (3.11)$$

which is identity on level 0, but which collapses the fiber of each level one piece \mathbb{F}_k , $k \geq 1$.

Remark 3.5 The precise identifications required to construct this building are also described in the Appendix. It is easy to see that both F_k and $D_{k+1,\infty}$ are normal crossings divisors, so in fact what we identify is their normalizations, via a canonical map (A.6.). Furthermore, their normal bundles are canonically dual to each other, see (A.7.), and the \mathbb{C}^* action on N induces an anti-diagonal \mathbb{C}^* action in the normal bundle

$$N_{F_k} \oplus N_{D_{k+1,\infty}}$$

of each component $F_k = D_{k+1,\infty}$ of the singular divisor W_1 of X_1 , where $k \geq 0$.

Example 3.6 Assume X has 4 real dimensions, and that the normal crossings divisor

$$V = V_1 \bigcup_{p_1=p_2} V_2$$

is the union of two submanifolds V_1 and V_2 intersecting only in a point $p = p_1 = p_2$. After rescaling once, we get 3 main pieces X , \mathbb{F}_1 and \mathbb{F}_2 together this an attaching map, see the

left hand side of Figure 1. Here \mathbb{F}_1 is a \mathbb{P}^1 bundle over $\widetilde{V}^1 = V_1 \sqcup V_2$, while \mathbb{F}_2 is just $\mathbb{P}^1 \times \mathbb{P}^1$ (over the point $\widetilde{V}^2 = p$). The divisor $D_{1,\infty} \subset \mathbb{F}_1$ is a copy of $\widetilde{V} = V_1 \sqcup V_2$ and it is attached to $F_0 = V = V_1 \cup_{p_1=p_2} V_2 \subset X$. Similarly, $D_{2,\infty} \subset \mathbb{F}_2$ is $\mathbb{P}^1 \times \infty \cup \infty \times \mathbb{P}^1$, and it is attached to F_1 , which is the disjoint union of two fibers of \mathbb{F}_1 over the points p_1 and p_2 in $V_1 \sqcup V_2$. Note that \mathbb{F}_1 does not descend as a bundle over V : the two fibers of \mathbb{F}_1 over singular locus p_1 and p_2 are not identified with each other, but rather each gets identified with different of fibers of \mathbb{F}_2 .

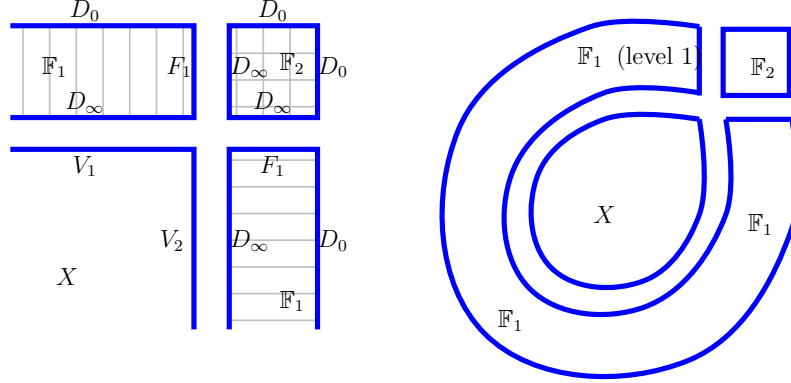


Figure 1: The pieces of a level 1 building

Example 3.7 Assume X has 4 real dimensions, but the normal crossings divisor V has only one component, self intersecting itself in just a point p . Locally, the situation looks just like the one in Example 3.6, with V having two local branches meeting at p . The only difference is that globally now \widetilde{V} has only one connected component containing both points p_1 and p_2 , see right hand side of Figure 1.

Remark 3.8 In the discussion above, we had a rescaling parameter λ normal to V , which means that we considered the action of $\lambda \in \mathbb{C}^*$ on the normal bundle N over \widetilde{V} . If \widetilde{V} has several connected components

$$\widetilde{V} = \bigsqcup_{c \in C} \widetilde{V}_c \quad (3.12)$$

then we could independently rescale normal to each one of them; this gives a $\lambda \in (\mathbb{C}^*)^C$ action, rather than just the diagonal one we considered before. Rescaling in all these independent directions now gives a multi-building, where each floor has a level associated to each connected component of \widetilde{V} . We could talk about a floor which is on level one normal to some of the components, but level zero normal to other components.

By iterating the rescaling process, we obtain level m buildings where we rescale m times normal to V , or more generally multi-buildings with m_c levels in the normal direction to each connected component \widetilde{V}_c of \widetilde{V} .

Definition 3.9 A level m building is a singular space X_m with a singular divisor V_m , called the zero section, that is obtained recursively from (X_{m-1}, V_{m-1}) by iterating the level one building procedure. In particular, a level m building comes with a resolution $(\widetilde{X}_m, \widetilde{D}_m)$ and an attaching map $\xi : (\widetilde{X}_m, \widetilde{D}_m) \rightarrow (X_m, D_m)$ that attaches all the floors together producing

the singular locus W_m of X_m , where $D_m = W_m \cup V_m$ is the total divisor. It also comes with a collapsing map

$$p_m : (X_m, V_m) \rightarrow (X_{m-1}, V_{m-1})$$

that collapses the level m floor or more generally a collapsing map that collapses any subset J of the levels $\{1, \dots, m\}$.

For example,

$$p : (X_m, V_m) \rightarrow (X_0, V_0) \tag{3.13}$$

collapses all positive levels (but leaves the level zero unaffected).

Note that as we add floors, the building grows bigger in several (local) directions. Starting with (X_{m-1}, V_{m-1}) , we then construct depth k pieces

$$\mathbb{F}_{k,m} = \mathbb{F}_k(V_{m-1}).$$

on level m . Since the divisor V_{m-1} has several pieces (it is itself a rescaled version of V_0), the number of depth $k \geq 2$ pieces increases (at least locally) as the building grows new levels.

Remark 3.10 Note that we used **depth** to measure how many local branches of V meet at a point. On the other hand, as a result of rescaling, we now also get **levels**, which measure how many times the target has been rescaled.

Example 3.11 Consider the simplest model of a level m building, the m times rescaled disk

$$(D^2)_m = D^2 \bigcup_{0=\infty} \mathbb{P}^1 \bigcup_{0=\infty} \dots \bigcup_{0=\infty} \mathbb{P}^1 \tag{3.14}$$

on which in some sense all the other level m buildings are modeled on. Its resolution has $m + 1$ components, each indexed by a level $l = 0, \dots, m$ with D^2 on level zero (unrescaled). The total divisor also has $m + 1$ components, each similarly labeled by a level $l = 0, \dots, m$ with the $0 \in D^2$ on level one. This defines a lower-semicontinuous level map l on $(D^2)_m$. It is discontinuous precisely at points y in the singular divisor, where it has two limits, indexing the two lifts y_{\pm} of y to consecutive levels, where $y_+ = 0$ is on the same level as y while $y_- = \infty$ is on the next level. Each point on the resolution comes with a sign $\varepsilon = \pm$ or 0 , keeping track of whether its coordinate is ∞ , 0 or neither. Then $\varepsilon = 0$ corresponds to a smooth point of $(D^2)_m$, while $\varepsilon = \pm$ corresponds to a point in the resolution of the total divisor. Intrinsically, ε keeps track of the weight of the \mathbb{C}^* action on each piece.

Example 3.12 Suppose next we are in the situation of Example 3.6. The first level had three components: two are \mathbb{P}^1 bundles over V_1 and respectively V_2 and another is a $\mathbb{P}^1 \times \mathbb{P}^1$ over a point. A second level would have five components. Two of them are still \mathbb{P}^1 bundles over V_1 and V_2 respectively, but there are now three copies of $\mathbb{P}^1 \times \mathbb{P}^1$. The way they come about is as follows. The zero divisor $V_{(1)}$ of the first floor consists now of 4 pieces: V_1 , V_2 but also two \mathbb{P}^1 's intersecting in a point p^1 (coming from the zero section of depth two piece $\mathbb{F}_{2,1}$ on the first floor). When we rescale again to get the second floor, $\mathbb{F}_{2,2}$ still a $\mathbb{P}^1 \times \mathbb{P}^1$ over the point p^1 , but $\mathbb{F}_{1,2}$ is now larger, it is a \mathbb{P}^1 bundle over $(V_1 \sqcup \mathbb{P}^1) \sqcup (V_2 \sqcup \mathbb{P}^1)$, so it really has four pieces: a \mathbb{P}^1 bundle over V_1 and respectively V_2 , as was the case in level one, but also two other $\mathbb{P}^1 \times \mathbb{P}^1$ pieces coming from rescaling over the two \mathbb{P}^1 fibers in $V_{(1)}$. All together this level 2 building has 4 copies of $\mathbb{P}^1 \times \mathbb{P}^1$; in general, such level m building will have m^2 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, and m copies of the \mathbb{P}^1 bundle over $V_1 \sqcup V_2$.

\mathbb{F}_1 level 2	\mathbb{F}_2	\mathbb{F}_2
\mathbb{F}_1 level 1	\mathbb{F}_2	\mathbb{F}_2
X , level 0	\mathbb{F}_1 level 1	\mathbb{F}_1 level 2

Figure 2: A level 2 building

Example 3.13 Consider again the situation in Example 3.6. Because $\tilde{V} = V_1 \sqcup V_2$ has two connected components, we can also independently rescale normal to each of them, getting instead a multi-building. For example, a level (2,2) multibuilding in this case looks just like the level 2 building described above, except that the pieces might be on different levels when regarded in different directions. For example, the 4 pieces $\mathbb{P}^1 \times \mathbb{P}^1$ now land one on each level (i, j) for $i, j = 1$ or 2 , while before one of them (the one on level (1,1)) landed on level 1, while the other three landed on level 2. More generally, the level m building from the example above can be regarded as a level (m, m) multibuilding when we rescale independently in the two directions, with exactly one copy of $\mathbb{P}^1 \times \mathbb{P}^1$ on each level (i, j) for $i, j = 1, \dots, m$. But in this case we can also have a multi-building with different number of levels in each direction, for example we could have just one level normal to V_1 but three normal to V_2 .

Example 3.14 Consider now the situation of Example 3.7. Locally everything looks the same as in Example 3.12 and even near p we have two independent local directions in which we could rescale as in Example 3.13. However, because \tilde{V} is now connected, globally there is only one scaling parameter $\lambda \in \mathbb{C}^*$ normal to V , so the two local scaling parameters at p have to be related to each other (they are essentially equal).

Remark 3.15 In general, for any level m building X_m , even if globally we may not be able to have separate directions of rescaling, at least semi-locally the picture always looks like part of a multilevel building. More precisely, fix a depth k point p in V , and locally index by $i \in I$ the k branches of V coming together at p . A neighborhood U_p of fiber F_p over p of the collapsing map $X_m \rightarrow X$ of (3.13) is a product of a small neighborhood O_p of p in the depth k stratum and k copies of an m -times rescaled disk $(D^2)_m$ of (3.14), one factor for each one of the k branches of V at p . This describes not only the tower of $(m+1)^k$ pieces of the resolution \tilde{X}_m with its total divisor \tilde{D}_m , but also their attaching map, just as in Example 3.11, except that now we have k directions to keep track of instead of just one.

In particular, each point $y \in X_m$ (which projects to p under the collapsing map $X_m \rightarrow X$) comes with a multi-level map

$$l : I \rightarrow \{0, \dots, m\} \quad (3.15)$$

as if it was part of a multilevel building (with k independent directions), where we would separately keep track of the level l_i for each one of the directions $i \in I$. Of course, as part of X_m this piece is on level $l = \max_i l_i$, but it appears as part of each one of the local levels l_1, \dots, l_k . Furthermore, a point \tilde{y} in the resolution \tilde{X}_m comes not only with the multi-level map (3.15) of its image in X_m , but also a multi-sign map

$$\varepsilon : I \rightarrow \{0, \pm 1\} \quad (3.16)$$

that keeps track of whether it is equal to zero, infinity or neither in the i 'th direction. This allows us for example to keep track of the various branches of the total divisor, and the fact that they come in dual pairs indexed by opposite multi-sign maps ε ; note that in general we can have a point which is in the zero divisor in some of the directions, on infinity divisor in other directions, and then in some other directions on neither, and this is precisely what ε records.

Equivalently, over a depth k point p of V , we can instead index each local piece P of the resolution \tilde{X}_m by a

$$\text{partition of } I \text{ into } I_0, \dots, I_m. \quad (3.17)$$

The local piece $P(I_0, \dots, I_m)$ is on level l with respect to the branches indexed by I_l , and thus can be thought as obtained by rescaling l times X near p in the I_l directions; for example, when $I_0 = I$, that piece is completely unrescaled and corresponds to a piece of the level zero X . More generally, for any local piece indexed by such a partition, the directions in I_0 are unrescaled, while the remaining j directions are rescaled (at least once). Then

$$I_0 \text{ determines a unique lift } p_0 \text{ of the point } p \text{ to the resolution } \widetilde{V}^j \quad (3.18)$$

of the depth j stratum of V while I_1, \dots, I_m index the directions of the

$$\text{fibers } (\mathbb{P}^1)^{j_1} \times \dots \times (\mathbb{P}^1)^{j_m} = (\mathbb{P}^1)^j \text{ of } \mathbb{F}_j \text{ over } \widetilde{V}^j. \quad (3.19)$$

This identifies the local piece $P(I_0, \dots, I_m)$ with a neighborhood in \mathbb{F}_j of the fiber over p_0 . Keeping track of which coordinates in the fiber of \mathbb{F}_j are infinity, zero or neither (which induces a further partition of I into I^\pm and I^0) then allows us to index any open stratum of the total divisor \tilde{D}_m in \tilde{X}_m over the point p by a

$$\text{partition of } I \text{ into } I_j^\pm, I_j^0 \text{ for } j = 0, \dots, m. \quad (3.20)$$

Then I^+ records the branches of the zero and fiber divisor, while I^- those of the infinity divisor.

Remark 3.16 The \mathbb{C}^* action of N_V induces a $(\mathbb{C}^*)^m$ action on a level m building such that each factor $\alpha_l \in \mathbb{C}^*$ rescales the level $l \geq 1$ piece of X_m normal to its zero divisor, fixes the level zero pointwise, and preserves the total divisor and its stratification, but not pointwise. As before, this is modeled by the $(\mathbb{C}^*)^m$ action on the rescaled disk $(D^2)_m$ in which each factor $\alpha_l \in \mathbb{C}^*$ acts on the \mathbb{P}^1 component of $(D^2)_m$ in level $l \geq 1$, but now extended as the diagonal action to the product of the disks in each normal direction to V . More precisely, each component of the resolution X_m whose (local) multi-levels are l_1, \dots, l_k , is acted by α_{l_i} on its \mathbb{P}^1 factor in direction i for each $l_i \geq 1$.

3.3 Local model near the divisor

Rescaling X by a factor of $\lambda \neq 0$ along a normal crossings divisor V similarly gives rise to manifold we denote X_λ and an identification

$$R_\lambda : X \rightarrow X_\lambda \quad (3.21)$$

exactly as in (3.5). Just as described in §3.1, the manifold X_λ has two regions, one is the complement of the $|\sqrt{\lambda}|$ -tubular neighborhood U_λ of V in X , on which R_λ is the identity, and

the other one is identified with O_λ , the complement of the $|\sqrt{\lambda}|$ -tubular neighborhood of the singular divisor W_1 in \mathbb{F}_V . The only difference now is that we have several overlapping local models, coming from the stratification of V .

Still, this perspective allows us to think of the rescaled X as a sequence of manifolds X_λ with varying $J_\lambda = R_\lambda^* J$, which as $\lambda \rightarrow 0$ converge to a level one building X_1 . In fact, X_λ can be thought as some sort of iterated symplectic sum: fix a level one building X_1 as in Definition 3.4, with appropriate identifications along corresponding divisors, including fixed isomorphisms (A.7.). For each (small) gluing parameter $\lambda \in \mathbb{C}^*$ we get the 'symplectic sum' X_λ (diffeomorphic to X , but with a deformed symplectic form) and such that X_λ converges to X_1 as $\lambda \rightarrow 0$. This approach is mentioned in Remark 7.7 of [IP1] and expanded on in [IP2]; it turns out for this paper to be more convenient than the approach of section 6 of [IP1].

We next describe the rescaling procedure in more detail. We will work in regions which are obtained from neighborhoods of depth k strata of V after removing neighborhoods of the higher depth stratum, where we have nice local models. Denote by U_δ the δ -tubular neighborhood of V in X , that is the image under ι of \tilde{U}_δ , the δ -disk bundle in the normal bundle model $N_V \rightarrow \tilde{V}$ over the resolution of V , and by

$$A(r, R) = \iota(\tilde{U}_R \setminus \tilde{U}_r) \quad (3.22)$$

the union of annular regions about V in X , which will give rise to the necks. Note that the region A still intersects V around depth $k \geq 2$ points, so it is very different from the region $U_R \setminus U_r$, an important distinction that will become relevant later in the paper.

The neighborhood U_δ has subregions U_δ^k corresponding to a product neighborhood of the depth k stratum V^k , over which ι is k -to-1. Each region U_δ^k can be then regarded as a subset of \mathbb{F}_k , where it is identified, after fiberwise rescaling by λ with the complement of a neighborhood of the divisor $F_k \cup D_{k,\infty}$ in \mathbb{F}_k to obtain X_λ .

More precisely, assume p is a depth k point of V (but away from the higher depth strata); locally index the k branches I of V coming together at p , and choose local coordinates u_1, \dots, u_k normal to each one of these branches (so in particular the j 'th branch is given by $u_j = 0$) such that the δ -tubular neighborhood U_δ of V is given in these coordinates by $|u_i| \leq \delta$ for some $i \in I$. The region U_δ^k is then described in these coordinates by

$$|u_i| \leq \delta \text{ for all } i = 1, \dots, k \quad (3.23)$$

as long as $p \in V^k$ but outside the δ neighborhood of the higher depth stratum V^{k+1} . This means that under the change of coordinates $v_i = \lambda/u_i$ for each $i \in I$, the region U_δ^k is identified with a region $O_\delta(\lambda)$ in \mathbb{F}_k described by the equations

$$|v_i| \geq |\lambda|\delta^{-1} \text{ for all } i = 1, \dots, k \quad (3.24)$$

where $p \in V^k$ is outside the δ neighborhood of the higher depth stratum. This region $O_\delta(\lambda)$ is nothing but complement of the union of tubular neighborhoods of the infinity divisor $D_{k,\infty}$ and the fiber divisor F_k in \mathbb{F}_k , see (3.7). For $\delta = |\sqrt{\lambda}|$ this change of coordinates therefore provides an identification R_λ of the $|\sqrt{\lambda}|$ -tubular neighborhood of V in X (broken into pieces as above) and the complement of the $|\sqrt{\lambda}|$ -tubular neighborhood of the singular divisor W_1 in \mathbb{F}_V , exactly as suggested by the iterated symplectic sum construction.

Therefore the (semi)-local model of X_λ over such a point p is given by the locus of the equations

$$u_i v_i = \lambda \quad \text{where} \quad |u_i| \leq \delta \quad \text{for all } i \in I \quad (3.25)$$

where u_i is a coordinate in N_i , the normal bundle to the i 'th local branch of V and v_i is the dual coordinate in the dual bundle N_i^* (which is allowed to equal infinity). Note that these equations are invariant under reordering of the branches, so they describe an intrinsic subset of $N_{V^k} \times \mathbb{F}_k$ where our semi-local analysis will take place.

The δ -neck $N_\lambda(\delta)$ of X_λ is the region in the above coordinates where

$$|u_i| < \delta \text{ and } |v_i| < \delta \text{ for some } i \quad (3.26)$$

and it globally corresponds to the annular region $A(|\lambda|\delta^{-1}, \delta)$ around V in X . The *upper hemisphere region* $H_\lambda(\delta)$ of X_λ corresponds to the region $A(|\lambda|, \delta)$ in X ; in coordinates it is described by:

$$|u_i| < \delta \text{ and } |v_i| < 1 \text{ for some } i \quad (3.27)$$

As $\lambda \rightarrow 0$, X_λ converges to a level one building X_1 ; for λ sufficiently small, the part of X_λ outside the δ -neck is canonically identified with the complement a certain neighborhood of W_1 in X_1 ; as both $\lambda, \delta \rightarrow 0$, this neighborhood expands to $X_1 \setminus W_1$. The upper hemisphere region $H_\lambda(\delta)$ similarly converge to the upper hemisphere of \mathbb{F}_V . The neighborhoods $U_\delta^k \setminus U_\delta^{k+1}$ all fit inside $N_{V^k} \times \mathbb{F}_k$ where they are described by the equations (3.25). In particular, over a point p as above, the level one building X_1 is described by the locus of the equations:

$$u_1 v_1 = 0, \quad \dots, \quad u_k v_k = 0 \quad (3.28)$$

where $|u_i| \leq \delta$ for all i , regarded as an intrinsic subset of $N_{V^k} \times \mathbb{F}_k$ (or more precisely inside the pullback of \mathbb{F}_k over N_{V^k}).

These describe the 2^k local pieces of X_1 coming together at p along the singular locus W_1 : each piece of X_1 is described by the vanishing of exactly k coordinates, but some of them may be $u_i \in N_i$ in which case the rest are the complementary indexed ones $v_j \in N_j^*$. Furthermore, the divisor W_1 has several local branches, each one described by the further vanishing of one of the remaining coordinates u_i or v_i , matching the description from Remark 3.15.

Remark 3.17 Of course, the rescaling procedure can be iterated finitely many times: start with X , rescale it by λ_1 , then rescale again the resulting manifold by λ_2 , etc. The limit as all $\lambda_a \rightarrow 0$ is then a level m building X_m , where we have similar semi-local models described over a depth k point p as intrinsic subsets of $(\mathbb{F}_k \times \mathbb{F}_k)^m$, modeled in each direction i by the process of m times rescaling a disk at the origin, see Remark 3.15. Therefore one way to describe this iterated rescaling is to start by choosing coordinates $u_{i,0} = u_i$ on X normal to V at p , together with m other sets of dual normal coordinates:

$$u_{i,l} \in N_i \cup \infty \text{ and } v_{i,l} \in N_i^* \cup \infty \text{ with } v_{i,l} = u_{i,l}^{-1} \quad (3.29)$$

for all $l = 1, \dots, m$ and all $i \in I$. These provide semi-local coordinates on X_m in a neighborhood of the fiber F_p over p of the collapsing map $X_m \rightarrow X$. Rescaling X means we

$$\text{make a change of coordinates } u_{i,l-1} v_{i,l} = \lambda_l \text{ at step } l, \quad (3.30)$$

for each $l = 0, \dots, m$ and all $i \in I$.

4 The refined limit of a sequence of maps

Consider now a sequence $\{f_n\}$ of maps in $\mathcal{M}_s(X, V)$ whose limit (in the usual stable map compactification) has some components in V and thus is not in $\mathcal{M}_s(X, V)$. We will describe how to rescale X once around V and construct a refined limit f into a level one building X_1 which has no nontrivial components in the singular locus, and fewer nontrivial components in the zero divisor. This will be the key main step in the inductive rescaling procedure used in the next section to construct the relatively stable map compactification.

We also want to describe in more details the collection of such possible limits f . In the case V is a smooth divisor, we proved in [IP2] that the limit satisfies a matching condition along the singular divisor. The arguments used there are semi-local (that is they involve a further analysis of what happens only in a tubular neighborhood of V), and extend to the case when V is a normal crossings divisor by similarly working in neighborhoods of each depth k stratum of V , where we have k different normal directions to V . In the case the limit f has no trivial components in W_1 , we will also get a matching condition along W_1 . First of all, we will see that similarly $f^{-1}(W_1)$ consists of nodes of the domain. The naive condition is that at each such node, f has matching intersection points with W_1 , including the multiplicities, as was the case in [IP1]. But it turns out that at points of depth $k \geq 2$, this is not enough to define relative GW invariants, and needs to be further refined.

First of all, according to our conventions which match those of [IP2], every map $f : \Sigma \rightarrow X$, will be regarded as a map $(f, \varphi) : \Sigma \rightarrow X \times \overline{\mathcal{U}}_{g,n}$, where the second factor is the universal curve of the domains. The energy of f in a region N of the target is then defined to be

$$E(f; N) = \frac{1}{2} \int_{f^{-1}(N)} |df|^2 + |d\varphi|^2 \quad (4.31)$$

Denote by α_V the minimum quanta of energy that a stable holomorphic map into V has. Note that according to our conventions, the energy also includes the energy of domain Σ , which is stable when $f_*[\Sigma] = 0$, and therefore $\alpha_V > 0$.

The key idea of Chapter 3 of [IP2] was that when V is smooth and $\delta > 0$ small, the limits of maps which have energy at most $\alpha_V/2$ in the δ -neck cannot have any components in the singular locus $V = V_\infty$ and thus have well defined leading coefficients along $V = V_\infty$ which are furthermore uniformly bounded away from 0 and infinity (the particular bound depends on δ and the choice of metrics).

In the case when V is a normal crossings divisor, it will no longer be true that the limit of maps which have small energy in the neck has no components in the singular locus. Now we could have some components in level one whose energy is smaller than $\alpha_V/2$: these are stable maps into \mathbb{F}_V , but whose projection to V does not have a stable model. In the case V was smooth, these correspond precisely to what we called trivial components in Definition 11.1 of [IP2]. When V is a normal crossings divisor, the appropriate extension of that definition is then the following:

Definition 4.1 *A trivial component in \mathbb{F}_k is a nonconstant holomorphic map $f : (\mathbb{P}^1, 0, \infty) \rightarrow (\mathbb{F}_k, D_0 \cup D_\infty)$ whose image lands in a fiber $(\mathbb{P}^1)^k$ of \mathbb{F}_k . In coordinates*

$$f(z) = (a_1 z^{s_1}, \dots, a_k z^{s_k})$$

where some, but not all of the a_i could be zero or infinity.

This is the only kind of stable map into $(\mathbb{F}_k, D_0 \cup D_\infty \cup F_k)$ which does not have a stable model when projected into D_0 : its domain is an unstable sphere with just two marked points

and thus its projection has energy zero. Note that as a result of the rescaling process, we have uniform control only on the energy of the projection, as the area of the fiber of \mathbb{F}_k could be arbitrarily small. We are now ready to state the first rescaling result:

Proposition 4.2 *Consider $\{f_n : C_n \rightarrow X\}$ a sequence of maps in $\mathcal{M}_s(X, V)$ which converge, in the usual stable map compactification, to a limit $f_0 : C_0 \rightarrow X$ which has at least one component in V .*

Then there exists a sequence $\lambda_n \rightarrow 0$ of rescaling parameters such that after passing to a subsequence, $R_{\lambda_n} f_n$ have a refined limit $g : C \rightarrow X_1$ with the following properties:

(a) *g is a map into a level one building X_1 refining f_0 :*

$$\begin{array}{ccc} C & \xrightarrow{g} & X_1 \\ \downarrow \text{st} & & \downarrow p \\ C_0 & \xrightarrow{f_0} & X \end{array} \quad (4.32)$$

(b) *g has no nontrivial components in the singular divisor W_1 ;*

(c) *g has at least one nontrivial component in level one which does not lie entirely in V_1 , and thus has fewer nontrivial components in the zero divisor V_1 compared with f_0 ;*

Any refined limit g that has properties (a)-(c) is unique up to rescaling the level one of the building by an overall factor $\lambda \in \mathbb{C}^$.*

Proof. In the case V is a smooth divisor, we proved this in Section 6 of [IP1] (see also Section 3 of [IP2]). The arguments used there to construct the refined limit are semi-local, in a neighborhood of V , and if set up right, easily extend to the case when V is a normal crossings divisor. The main rescaling argument consists of two parts, first the construction of the refined limit by rescaling the target near V , and later on a further analysis of the properties of this refined limit. For the first part of the argument, we work separately in neighborhoods of depth k stratum V^k but away from the higher depth strata; for the second part of the argument, we work in the necks, where the transition between these local models happens. As this is one of the crucial steps in the construction of the relatively stable map compactification, we include below the complete details of both of these arguments.

Step 0. Preliminary considerations. Assume for simplicity that the domains C_n are smooth (otherwise work separately on each of their components) and that the original limit C_0 is a stable curve (see Remark 2.1).

Around each point $x \in C_0$, denote by $B(x, \varepsilon)$ the ball about x of radius ε in the universal curve, and by $B_n(x, \varepsilon)$ its intersection with C_n . Around each node x of C_0 we can choose local coordinates z, w on the universal curve such that the domains C_n are described by

$$zw = \mu_n(x) \quad (4.33)$$

where $\mu_0(x) = 0$. Denote by D the collection of nodes of C_0 , and by \tilde{C}_0 the resolution of C_0 . For each point $x \in \tilde{C}_0$, denote by $\gamma_\varepsilon(x)$ the oriented boundary of the ε -disk about x in \tilde{C}_0 , and by $\gamma_{n, \varepsilon}(x)$ the corresponding boundary component of $B_n(x, \varepsilon)$.

Denote by $P_n \subset C_n$ the collection of marked points of f_n , which include all the points in $f_n^{-1}(V)$, with their contact information recorded by s . As $n \rightarrow \infty$, they converge to the marked points P_0 of C_0 , but the original limit f_0 has only a partially defined contact information to V , which we describe next. Each component Σ of C_0 has a depth $k(\Sigma)$, defined as the maximum k such that $f_0(\Sigma) \subset V^k$. The restriction of f_0 to Σ has a well defined contact multiplicity to the next stratum V^{k+1} ; each such contact point has an associated contact

information (2.3) along V^{k+1} , including its depth $k(x) > k$ (with respect to the stratification of V). For each $j > k$, will denote by $R_k^j \subset \tilde{C}_0$ the collection of contact points of depth j that belong to a component of depth k . Denote by R the union of all these contact points, and by R_k those that have depth k while by R^j those that belong to a depth j component. Also denote by $f^k : C^k \rightarrow V^k$ the restriction of f_0 to C^k , the part of C_0 consisting of depth k components; in fact, $f^k \in \mathcal{M}_{s^k}(\tilde{V}^k, \tilde{V}^{k+1})$ and

$$f_0^{-1}(V^k \setminus V^{k+1}) = (C^k \setminus R^k) \sqcup R_k \quad (4.34)$$

This allows us to break-up the domain of C_0 and C_n into pieces whose image lies in controlled regions where the rescaling happens (more precisely in a neighborhood of V^k , but away from a neighborhood of the higher depth stratum V^{k+1}). Recall that U_δ denotes the δ -tubular neighborhood of V , and U_δ^k denotes the corresponding neighborhood of V^k , see (3.23).

The complement of the 'neck' regions $B(x, \varepsilon)$ about all points $x \in R$ (for $\varepsilon(x) > 0$ sufficiently small) decomposes C_n into pieces $(C_n^k)'$, which limit as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ to $C^k \setminus R^k$. We will denote by C_n^k the union of the $(C_n^k)'$ piece together with all the 'neck' pieces around points in R_k . The boundary of C_n^k then decomposes as $\partial_+ \sqcup -\partial_-$ where

$$\partial_+ C_n^k = \bigsqcup_{x \in R_k} \gamma_{n, \varepsilon}(x) \quad \text{while} \quad \partial_- C_n^k = \bigsqcup_{x \in R^k} \gamma_{n, \varepsilon}(x) \quad (4.35)$$

The curve C_n is obtained by joining C_n^k along $\partial_- C_n^k$ and $\partial_+ C_n^k$ to lower and respectively higher depth pieces C_n^j .

By construction and (4.34), f_0 maps C_0^k in some neighborhood $U_{M(\varepsilon)/2}^k$ of the depth k stratum but away from some much smaller neighborhood $U_{2m(\varepsilon)}^{k+1}$ of the higher depth stratum, such that the incoming part $f_0(\partial_- C_0^k)$ of the boundary ends up outside the neighborhood $U_{2m(\varepsilon)}^k$ of the depth k stratum, while the outgoing part $f_0(\partial_+ C_0^k)$ of the boundary ends up inside the $U_{M(\varepsilon)/2}^k$ neighborhood of the depth $k+1$ stratum. The sizes of these tubular neighborhoods depend on f_0 , but can be chosen uniform in k and also such that $M(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Fix $0 < \delta \ll 1$ small enough so that, for all $k \geq 0$ the depth k piece f^k of the original limit f_0 has very small energy in a δ -neighborhood of the higher depth stratum V^{k+1} :

$$\text{the energy of } f^k \text{ in the region } U_\delta^{k+1} \text{ is less than } \alpha_V/100 \quad (4.36)$$

Of course, we have assumed that f_0 has at least one component in V thus the energy of the restriction of f_0 to the complement of C^0 is at least α_V . Assume $\varepsilon > 0$ sufficiently small so that $2M(\varepsilon) < \delta$. Since $f_n \rightarrow f_0$ in the stable map compactification as maps into X , for n sufficiently large,

- (i) $f_n(C_n^k)$ lies in $U_M^k \setminus U_m^{k+1}$;
- (ii) $f_n(\partial_- C_n^k)$ lies $U_M^k \setminus U_m^k$ while $f_n(\partial_+ C_n^k)$ lies in $U_M^{k+1} \setminus U_m^{k+1}$;
- (iii) the restriction of f_n to C_n^k has energy at most $\alpha_V/50$ in the region U_δ^{k+1} ;
- (iv) the restriction of f_n to the complement C_n^0 has energy at least $2\alpha_V/3$.

Step 1. Constructing the refined limit. Next we rescale X around V by a certain amount λ_n to catch a refined limit of the sequence f_n . To find λ_n , we consider the energy $E(t)$ of f_n inside the annular region $A(t, \delta)$ of (3.22) around V in X . Because f_n was assumed to be smooth, the energy $E(t)$ is a continuously decreasing function in t , which is equal to 0 when $t = \delta$ and is at least $2\alpha_V/3$ when $t = 0$ (because the restriction of f_n to the complement

of C_n^0 lies in the U_δ by (i) and has energy at least $2\alpha_V/3$ by (iv)). So for n sufficiently large, we can find a $\lambda_n \neq 0$ such that

$$\text{the energy of } f_n \text{ in the annular region } A(|\lambda_n|, \delta) \text{ is precisely } \alpha_V/2 \quad (4.37)$$

where $\delta > 0$ is fixed such that (4.36). Then $\lambda_n \rightarrow 0$, because if they were bounded below by $\mu > 0$ then in the limit the energy in the annular region $A(\mu, \delta)$ of f_0 and thus of f^0 would be $\alpha_V/2$ which contradicts (4.36).

As the rescaling maps $R_{\lambda_n} : X \rightarrow X_{\lambda_n}$ of (3.21) are identity outside the $|\lambda_n|^{1/2}$ neighborhood of V , the restriction of f_n to C_n^0 will be unaffected by rescaling and thus will uniformly converge (on compacts away from the nodes) to the restriction of f^0 to C_0^0 . Taking also $\varepsilon \rightarrow 0$, we get usual Gromov convergence of the restrictions of $R_{\lambda_n} f_n$ to C_n^0 to a limit g_0 , which is equal in this case to the original depth zero part of the limit $f^0 : C^0 \rightarrow X$. In particular, the restriction of $R_{\lambda_n} f_n$ to the outgoing part of the boundary $\partial_+ C_n^0$ uniformly converge to the restriction of g^0 to $\partial_+ C_0^0$, where we have a local model (2.3) around V .

Now proceed by induction on $k \geq 1$. Assume that for all $\varepsilon_n \rightarrow 0$, after possibly making the ε_n even smaller, the restrictions of $R_{\lambda_n} f_n$ to $\bigcup_{j < k} C_n^j(\varepsilon_n)$ converge to a limit $g : \bigcup_{j < k} B_j \rightarrow \bigcup_{j < k} \mathbb{F}_j$ which refines the depth less than k part of the original limit f_0 . This in particular means that the boundary of $\bigcup_{j < k} C_n^j$ converges to finitely many marked points of $\bigcup_{j < k} B_j$ and that the restrictions of $R_{\lambda_n} f_n$ to this boundary converge to the image under g of these points.

We want to extend this for the restrictions to $\bigcup_{j \leq k} C_n^j$. Note that by construction, $\bigcup_{j < k} C_n^j$ has only outgoing boundary, which matches the incoming one of $\bigcup_{j \geq k} C_n^j$. Therefore it suffices to only consider the restrictions to C_n^k and show that after possibly shrinking only the outgoing part of its boundary we get a convergent subsequence to some $g_k : B_k \rightarrow \mathbb{F}_k$ which refines $f^k : C^k \rightarrow V^k$. Then the restriction of sequence $R_{\lambda_n} f_n$ to $\bigcup_{j \leq k} C_n^j$ will automatically converge to a continuous limit $g : \bigcup_{j \leq k} B_j \rightarrow \bigcup_{j \leq k} \mathbb{F}_j$ with the desired properties.

By (i), the restrictions of f_n to C_n^k lie in the δ -neighborhood U_δ^k of V^k , but outside the $m(\varepsilon)$ -tubular neighborhood of the next stratum V^{k+1} ; after rescaling by λ_n , we can identify this region with the complement of the $|\lambda_n|\delta^{-1}$ neighborhood of the infinity divisor $D_{k,\infty}$ in \mathbb{F}_k and of the $m(\varepsilon)$ neighborhood of the fiber divisor F_k over the higher depth stratum, see (3.23) and (3.23).

Next consider the restriction of f_n to each piece $\gamma_{n,\varepsilon}(x)$ of the boundary of C_n^k , which after rescaling ends up in a corresponding annular region around $D_{k,\infty}$ and respectively F_k . We claim that the image of $\gamma_{n,\varepsilon}(x)$ under $R_{\lambda_n} f_n$ can be capped off with a small energy disk in \mathbb{F}_k around this divisor. For the incoming pieces of the boundary (i.e. for each $x \in R_j^k$ with $j < k$), this is because by induction $R_{\lambda_n}(\gamma_{n,\varepsilon}(x))$ uniformly converges as $n \rightarrow \infty$ to the restriction of $g(\gamma_\varepsilon(x))$ where we already have a local model coming from f^j for each $j < k$; in particular, the intersection of these capping disks with the infinity section is uniformly bounded. For the outgoing pieces of the boundary (i.e. for each $x \in R_k^j$ with $j > k$) we have the local model of f^k along V^{k+1} , to which the unrescaled $f_n(\gamma_{n,\varepsilon}(x))$ converges uniformly for fixed $\varepsilon > 0$. So after possibly choosing a smaller $\varepsilon_n(x) > 0$ (depending also on n for each such outgoing x), we can also cap off the image of $\gamma_{n,\varepsilon}(x)$ under $R_{\lambda_n} f_n$ by a small energy disk about the fiber divisor over the higher depth stratum (away from the infinity section).

The resulting homology class in \mathbb{F}_k of the capped surface is constant in n , because is determined by (i) the homology class of its projection onto the zero section which depends only on f_0 and (ii) the intersection with the infinity divisor, where we already have uniform control by induction. This means *a fortiori* that the restriction of the maps $R_{\lambda_n} f_n$ to C_n^k , which are $R_{\lambda_n}^*(J, \nu)$ holomorphic, have uniformly bounded energy in \mathbb{F}_k (where the symplectic area of the fibers is fixed, but small) and therefore have Gromov convergent subsequence to

a (J_0, ν_0) limit $g_k : B_k \rightarrow \mathbb{F}_k$, defined (after removing its singularities) on a closed, possibly nodal curve B_k which is obtained from C^k after possibly inserting some bubble trees. The convergence is uniform on compacts away from the nodes of B_k and in Hausdorff distance.

This constructs inductively a refined limit $g : C \rightarrow X_1$, defined from $C = \cup B_k$ into a level one building (the depth k is bounded by $\dim X$, so the process terminates in finitely many steps). By construction, the refined limit g has energy at most $\alpha_V/2$ around the infinity section of \mathbb{F}_V , and therefore has no nontrivial components there (as these would carry at least α_V energy), giving (b). If g had only trivial components in the first level but outside V_1 , then g would have very small energy in the upper-hemisphere of \mathbb{P}_V , contradicting the choice (4.37). Therefore the limit satisfies (c) as well. This concludes the proof of Lemma 4.2. \square

Step 2: Further properties of the refined limit. We next want to understand in more details the behavior of the refined limit $g : C \rightarrow X_1$ constructed by Lemma 4.2 around the singular divisor W_1 where the pieces of the building are joined together; for that we will restrict ourselves to the neck regions (both of the domain and of the target) where we will use the local models (4.33) on the domain and (3.25) in the target.

First of all, every component Σ of the refined limit $g : C \rightarrow X_1$ that does not land in W_1 has a lift $\tilde{g} : \Sigma \rightarrow \tilde{X}_1$ to the resolution \tilde{X}_1 , where it has a well defined order of contact with \tilde{W}_1 ; the components that land in V_1 have only a partial contact information to the higher depth strata of V_1 ; the components that land in W_1 must project to a point under $X_1 \rightarrow X$, and have only a partial contact information to the higher depth strata of $\tilde{W}_1 \cup V_1$. The contact information of g refines the partial contact information of the depth k part of the original limit f_0 to the higher depth stratum of V .

In this section we use the semi-local models for the target described in Remark 3.15, and in particular describe the strata of the resolution of the total divisor \tilde{D}_1 in terms of strata of V together with a multilevel and a multisign map. Fix a point $x \in \Sigma$ such that $g(x) \in W_1 \cup V_1$, and denote by p the image of $g(x)$ under the collapsing map $X_1 \rightarrow X$. The image $\tilde{g}(x)$, which is a point in $\tilde{W}_1 \cup V_1$, comes with both a multi-level map (3.15) and a sign map (3.16) that associate to each local branch i of V at p a local level $l_i(x) = 0$ or 1 and a sign $\varepsilon_i(x) = \pm 1, 0$ recording whether it is equal to infinity, zero or neither in that direction. Furthermore, \tilde{g} has an associated contact information to \tilde{W}_1 and a possibly partial one to V_1 at x . This information includes

$$\text{a partition of } I \text{ into } I^\pm(x), I^0(x) \text{ and } I^\infty(x). \quad (4.38)$$

For each $i \in I^\pm(x)$, $\varepsilon_i(x) = \pm$ (respectively) and we also have a well defined contact multiplicity $s_i(x) > 0$ to the total divisor and a leading coefficient $a_i(x) \neq 0$ which is now regarded as an element of

$$a_i(x) \in N_i^{\varepsilon_i(x)} \otimes (T_x^* \Sigma)^{s_i(x)} \quad \text{for all } i \in I^\pm(x) \quad (4.39)$$

The directions $i \in I^0(x)$ have $\varepsilon_i(x) = 0$ and correspond to contact of order zero with the total divisor in those directions, so we set $s_i(x) = 0$ while the leading coefficient is $a_i(x) \in N_i$, the coordinate of $\tilde{g}(x)$ in the i 'th direction (which is nonzero). Finally, the directions $i \in I^\infty(x)$ have $\varepsilon_i(x) = +$ and $l_i(x) = 1$ and correspond to those directions in which $g(\Sigma)$ is entirely contained in the zero divisor V_1 . These count as undefined or infinite order of contact $s_i(x) = \infty$ while their leading coefficient $a_i(x) = 0$. In local coordinates z on Σ at x , and normal coordinates $u_{i,l} \in N_i$ to the zero divisor in each level l , $g(z)$ then has an expansion

$$(u_{i,l})^{\varepsilon_i(x)} = a_i(x) z^{s_i(x)} + O(|z|^{s_i(x)}) \quad \text{for all } i \notin I^\infty(x) \quad (4.40)$$

Lemma 4.3 Consider a sequence $f_n \in \mathcal{M}_s(X, V)$ as in Lemma 4.2, and let $g : C \rightarrow X_1$ denote its refined limit constructed there. Then g has a lift to the resolution of the level one building, which comes with a (possibly partial) contact information s to $\widetilde{W}_1 \cup V_1$ as described above. If we denote by C' any intermediate curve $C \rightarrow C' \rightarrow C_0$, then all the contact points of g descend to special points of C' , and moreover:

- $s(x_-) = s(x_+)$ and $\varepsilon(x_-) = -\varepsilon(x_+)$ for each node $x_- = x_+$ of C' ;
- $s(x) = s(x_n)$ and $\varepsilon(x) = \varepsilon(x_n)$ for each marked point $x \in C'$ which is the limit of marked points x_n of C_n

(whenever both sides are defined). Furthermore any special fiber of $C \rightarrow C'$ or $C' \rightarrow C_0$ is a string of trivial components with two end points (broken cylinder).

Proof. First of all, we can decompose the domain C into two pieces, C^0 and B where C^0 consists of nontrivial components, and B consisting of components that get collapsed under the two maps $C \rightarrow C_0$ and $X_1 \rightarrow X$. Then each connected component B_i of B is an unstable genus zero curve (bubble tree) with either one or two marked points, corresponding to a special fiber of $C \rightarrow C_0$ over either:

- (a) a non special point in the case B_i has only one marked point
- (b) a node in the case B_i has two marked points and both belong to $C^0 \cap B$ or
- (c) a marked point in the case B_i has two marked points, but only one belongs to $C^0 \cap B$ while the other is a marked point of C .

We have the same description for the special fibers of $C \rightarrow C'$ and of $C' \rightarrow C_0$ for any intermediate curve $C \rightarrow C' \rightarrow C_0$.

For any point $x \in C_0$, choose local coordinates at x on the universal curve of the domains (containing C_0) and normal coordinates to V in the target X at p as above (where $p = f_0(x)$ is a depth $k \geq 0$ point of V). Using the notations of the proof of Lemma 4.2, for $\varepsilon, \delta > 0$ sufficiently small and n sufficiently large, the image under f_n of $B_n(x, \varepsilon)$ is mapped in the δ neighborhood of p in X but away from the δ neighborhood of the depth $k + 1$ stratum. Furthermore, since $f_n^{-1}(V) = P_n$ then f_n maps $B_n(x, \varepsilon) \setminus P_n$ into the annular region O_δ

$$0 < |u_i| < \delta \quad \text{for all } i \in I \quad (4.41)$$

of N_{V^*} around p (away from the higher depth stratum). This region is homotopic to $(S^1)^k$, one factor for each branch i of V at p .

If we also fix to begin with global coordinates on all bubble components in B , we have a similar story for any point x in any one of the intermediate curves C' : f_n takes a sufficiently small punctured neighborhood $B_n(x, \varepsilon) \setminus P_n$ into the annular region (4.41), where now $B_n(x, \varepsilon)$ denotes the intersection of C_n with the ball $B(x, \varepsilon)$ about x in the local model for domains containing their intermediate limit C' . Topologically, $B_n(x, \varepsilon)$ is either a disk or an annulus, depending whether x is a smooth point or a node of C' . In particular, for every lift \tilde{x} of x to the resolution of C' and thus to \tilde{C} , the corresponding boundary loop $f_n(\gamma_{n, \varepsilon}(\tilde{x}))$ has a well defined winding number $s_{i, n}(\tilde{x})$ about the branch i of V at p .

Furthermore, for $\varepsilon > 0$ sufficiently small, $B_n(x, \varepsilon)$ contains no points from P_n if x is not a marked point of C' and otherwise it contains precisely one point $x_n \in P_n$ which limits to x as $n \rightarrow \infty$. This implies that, for n sufficiently large, and all $i \in I$,

- (a) if x is not a special point of C' then the winding numbers $s_{i, n}(\tilde{x}) = 0$;
- (b) if $x_\pm \in \tilde{C}$ correspond to a node x of C' then $s_{i, n}(x_+) + s_{i, n}(x_-) = 0$.
- (c) if $x \in C'$ is the limit of the marked points $x_n \in C_n$ then $s_{i, n}(\tilde{x}) = s_i(x_n)$;

But the winding numbers $s_{i,n}(\tilde{x})$ of f_n are related to those of the refined limit g . This is simply because the winding numbers of f_n agree with those of the rescaled maps $R_{\lambda_n} f_n$, and these converge uniformly on compacts away from the nodes of C to the refined limit $g : C \rightarrow X_1$, which has well defined winding numbers about the zero section in all directions $i \notin I^\infty$. Since the loops $\gamma_{n,\varepsilon}(\tilde{x})$ stay away from all the nodes of C (for ε sufficiently small) then for n sufficiently large:

$$s_{i,n}(\tilde{x}) = \varepsilon_i(\tilde{x}) s_i(\tilde{x}) \text{ for all } i \notin I^\infty \quad (4.42)$$

because of the expansion (4.40) of g at \tilde{x} . Note that this give us no information about the winding numbers in the direction of I^∞ , where the winding numbers of g are undefined.

In particular, any contact point \tilde{x} of g with $W_1 \cup V_1$ or any of its strata has $s_i(\tilde{x}) > 0$ in some direction i which rules out case (a): if $\tilde{x} \in \tilde{C}$ descends to a non special point x of C' , then $s_{i,n}(\tilde{x}) = 0$ by (a) which contradicts (4.42).

In case (b), for any node x of C' then $s_{i,n}(x_-) + s_{i,n}(x_+) = 0$ and so (4.42) implies that

$$s_i(x_-) = s_i(x_+) \text{ and } \varepsilon_i(x_-) = -\varepsilon_i(x_+) \quad \text{for all } i \notin I^\infty(x) \stackrel{\text{def}}{=} I^\infty(x_1) \cup I^\infty(x_2)$$

as both sides are well defined for such an i . If x is the limit of contact points x_n of f_n to V then (4.42) implies that

$$s_i(x) = s_i(x_n) \text{ and } \varepsilon_i(x) = \varepsilon_i(x_n) \quad \text{for all } i \notin I^\infty(x)$$

So for example x is a contact point of g with the zero divisor V_1 if and only if x_n was one for f_n to V .

Finally, this discussion implies that there are no components of B with just one marked point (otherwise, contracting such component would give a curve C' and a non special point x on it which is impossible as case (a) is ruled out for all intermediate curves C'). Therefore all the components of B have precisely 2 special points, which must be the contact points with the zero and the infinity divisor in their fiber, and thus are indeed trivial components as in Definition 4.1. Furthermore, the only special fibers of $C \rightarrow C'$ or $C' \rightarrow C_0$ are strings of trivial components (broken cylinders). \square

Remark 4.4 Lemma 4.3 and the discussion preceding it shows that each node x of C_0 comes with a partition of the original indexing set I into $I^\infty(x)$, $I^0(x)$ and then $I(x)$. $I^\infty(x)$ records those directions in which at least one of the local branches of g lies in the total divisor W_1 while $I^0(x)$ records the directions in which the i 'th coordinates of both $\tilde{g}(x_\pm)$ are nonzero, so both branches stay away from the total divisor in those directions. The remaining directions $i \in I(x)$ come with a multiplicity $s_i(x) > 0$ and opposite signs $\varepsilon_i(x_+) = -\varepsilon_i(x_-) \neq 0$, recording the two opposite sides of the level one building from which the two branches of g come into the singular divisor, and also two leading coefficients $a_i(x_\pm) \neq 0$ which are naturally elements of

$$a_i(x_+) \in N_i^{\varepsilon_i(x_+)} \otimes (T_{x_+} C)^{s_i(x)} \quad \text{and} \quad a_i(x_-) \in N_i^{\varepsilon_i(x_-)} \otimes (T_{x_-} C)^{s_i(x)} \quad (4.43)$$

where N_i is the branch of the normal bundle to V indexed by $i \in I(x)$.

Example 4.5 In the situation of Example 3.6 we could have two nodes of the domain mapped to p , one between the components X and \mathbb{F}_2 while the other one between the two \mathbb{F}_1 components, but no node at p between say X and \mathbb{F}_1 (as the branches of g would not be on opposite sides of the singular divisor in all local directions). In the first case the node is between level 0 and level 1 (really local level (0,0) and (1,1)), while in the second case it is between two level 1 floors, or more accurately between a local level (0,1) and (1,0) floor.

Remark 4.6 As a consequence of Lemma 4.3, a refined limit $g : C \rightarrow X_1$ which has no components in the total divisor $D_1 = W_1 \cup V_1$ lifts, after labeling the nodes of C , to a unique map \tilde{g} into \tilde{X}_1 , the resolution of the level one building X_1 , with

$$\tilde{g} \in \mathcal{M}_{s_{\pm} \sqcup s}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1)$$

where $s_{\pm}(x)$ records the extra contact information $s(x_{\pm})$ at the pair of points x_{\pm} corresponding to a node $x \in C$. The domain \tilde{C} of \tilde{g} is a resolution of the domain C of f . The combined attaching map identifies pairs of marked points of \tilde{C} to produce the nodes of C , and simultaneously attaches the targets together to produce the singular locus W_1 . This describes the limit g in terms of its normalization $\tilde{g} : \tilde{C} \rightarrow \tilde{X}_1$.

The attaching map ξ that attaches the pieces of \tilde{X}_1 , when restricted to a level k stratum is a degree 2^k cover of W_1 . At each node $x_1 = x_2$ mapped into this stratum we also have a partition of the $2k$ normal directions $N_{V^k} \oplus N_{V^k}^*$ into two length k dual indexing sets $I_W(x_1)$ and $I_W(x_2)$ that record the two opposite local pieces of \tilde{X}_1 containing the two local branches of g at that node. According to our setup, this information is already encoded by s_{\pm} , as is the topological information about the domains and the homology of the images, see Remark 2.3. Note that C may have some usual nodes (i.e. not mapped into W_1), but then according to our conventions these give rise to marked points with depth $k = 0$, also recorded in s_{\pm} , see Remark 2.2.

Furthermore, the image of the normalization $\tilde{g} : \tilde{C} \rightarrow \tilde{X}_1$ under the evaluation map (2.7) at the pairs of marked points giving the nodes:

$$\text{ev}_{s_+} \times \text{ev}_{s_-} : \mathcal{M}_{s \sqcup s_{\pm}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) \longrightarrow (W_1)_{s_+} \times (W_1)_{s_-} \quad (4.44)$$

lands in the diagonal Δ . We will call these the *naive matching conditions* because when s_{\pm} contains depth ≥ 2 points, the dimension of this stratum is in general bigger than the dimension of $\mathcal{M}_s(X, V)$!

The refined limit g is only well defined up to an overall rescaling parameter $\lambda \in \mathbb{C}^*$ that acts by rescaling on the level one of the building and by construction the limit g has at least one component which is not fixed by this action. This \mathbb{C}^* action induces an action on the moduli space of maps into \tilde{X}_1 , and thus the normalization of such a limit is really in the inverse image of the diagonal Δ under the map

$$\text{ev}_{s_+} \times \text{ev}_{s_-} : \mathcal{M}_{s \sqcup s_{\pm}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) / \mathbb{C}^* \longrightarrow (W_1)_{s_+} \times (W_1)_{s_-} \quad (4.45)$$

We will include this information later on in this section. For now, let us notice that:

Lemma 4.7 *The difference between the expected dimensions of $\mathcal{M}_s(X, V)$ and that of the stratum $\text{ev}_{s_{\pm}}^{-1}(\Delta)$ of (4.44) is equal to*

$$\dim \mathcal{M}_s(X, V) - \dim \text{ev}_{s_{\pm}}^{-1}(\Delta) = 2 \sum_{x \in P(s_+)} (1 - k(x))$$

where $P(s_+)$ denotes the collection of marked points associated with the sequence s_+ .

Proof. This is a simple adaptation of the calculations of [IP2] to this context. The expected dimension of $\text{ev}_{s_{\pm}}^{-1}(\Delta)$ is

$$\dim \text{ev}_{s_{\pm}}^{-1}(\Delta) = \dim \mathcal{M}_{\tilde{s}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) - \dim (W_1)_{s_+}$$

where $\tilde{s} = s \sqcup s_{\pm}$. So the difference is

$$\dim \mathcal{M}_s(X, V) - \dim \text{ev}_{s_{\pm}}^{-1}(\Delta) = \dim \mathcal{M}_s(X, V) - \dim \mathcal{M}_{\tilde{s}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) + \dim (W_1)_{s_+}$$

where

$$\begin{aligned} \dim \mathcal{M}_s(X, V) &= 2c_1(TX)A_s + (\dim X - 6)\frac{\chi}{2} + 2\ell(s) - 2A_sV \\ \dim \mathcal{M}_{\tilde{s}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) &= 2c_1(T\tilde{X}_1)A_{\tilde{s}} + (\dim X - 6)\frac{\tilde{\chi}}{2} + 2\ell(\tilde{s}) - 2A_{\tilde{s}}\tilde{W}_1 - 2A_{\tilde{s}}V_1 \end{aligned}$$

But $\chi = \tilde{\chi} - 2\ell(s_+)$, $\ell(\tilde{s}) = \ell(s) + 2\ell(s_+)$ and $A_sV = |s| = A_{\tilde{s}}V_1$, while Lemma 2.4 of [IP2] adapted to this context gives

$$c_1(TX)A_s = c_1(T\tilde{X}_1)A_{\tilde{s}} - 2A_{\tilde{s}}\tilde{W}_1$$

Therefore

$$\dim \mathcal{M}_s(X, V) - \dim \mathcal{M}_{\tilde{s}}(\tilde{X}_1, \tilde{W}_1 \cup \tilde{V}_1) = (\dim X - 2)\ell(s_+)$$

On the other hand, since the image under the evaluation map of each depth k point lands in a codimension $2k$ stratum of X_1 then

$$\dim (W_1)_{s_+} = \sum_{x \in P(s_+)} (\dim X - 2k(x))$$

and thus the difference in dimensions is exactly as stated. \square

Even after dividing by the \mathbb{C}^* action on the moduli space, Lemma 4.7 still implies that if we want to construct a relatively stable map compactification in the case when the normal divisor is singular (has depth $k \geq 2$) pieces, then we would need some refined matching condition, otherwise the boundary stratum is larger dimensional than the interior. Luckily, the existence of such refined compactification follows after a careful examination of the arguments in [IP2].

It turns out that when $k \geq 2$, not all the maps $g : C \rightarrow X_1$ (without components in $W_1 \cup V_1$) whose resolution \tilde{g} satisfies the naive matching condition can occur as limits after rescaling of maps $f_n : C_n \rightarrow X$ in $\mathcal{M}_s(X, V)$. To describe those that occur as limits, we use the results of Section 5 of [IP2]. For each node x of C , we will work in the local models (4.33) on the domain and (3.25) in the target, using the local coordinates as described. The results of Lemmas 4.2 and 4.3 can then be strengthened as follows:

Lemma 4.8 *Consider $f_n : C_n \rightarrow X$ a sequence of maps in $\mathcal{M}_s(X, V)$ as in Lemma 4.2, and further assume that its refined limit $f : C \rightarrow X_1$ constructed there has no components in the total divisor D_1 . Then for each node $x_- = x_+$ of C , we have the following relation:*

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\mu_n(x)^{s_i(x)}} = a_i(x_-)a_i(x_+) \quad \text{for each } i \in I(x) \quad (4.46)$$

where $\mu_n(x)$ are the gluing parameters (4.33) describing C_n at x in terms of C , λ_n is the sequence of rescaling parameters in the target, while $s_i(x) > 0$ and $a_i(x_{\pm}) \neq 0$ are the contact multiplicities and respectively the leading coefficients of the expansion (4.40) of the refined limit f at x_{\pm} .

Proof. Note that according to our conventions, the condition (4.46) is vacuous ($I(x) = \emptyset$) unless the image of the node x is mapped to the singular locus. For each node x of C that is mapped to a depth $k(x) \geq 1$ stratum of W_1 with matching multiplicities $s(x)$, we will work in the local models described above, where we can separately project our sequence into each direction $i \in I(x)$, where the local model is that of rescaling a disk around the origin, as explained in more details in the next section. The projections are now maps into a rescaled family of disks, precisely the situation to which Lemma 5.3 of [IP2] applies to give (4.46) in each direction, after using the expansions (4.40). \square

Remark 4.9 The local model of $\mathcal{M}_s(X, V)$ near a limit point $f : C \rightarrow X_1$ which has no components in W_1 is described by tuples $(\tilde{f}, \mu, \lambda)$ satisfying an *enhanced matching condition* (4.47). Here $\tilde{f} : \tilde{C} \rightarrow \tilde{X}_1$ is an element of $\text{ev}_{s_{\pm}}^{-1}(\Delta) \subset \mathcal{M}_{s \sqcup s_{\pm}}(\tilde{X}, \tilde{W}_1)$ satisfying the naive matching condition along W_1 described in Remark 4.6, $\lambda \in \mathbb{C}$ is the gluing (rescaling) parameter of the target, and $\mu \in \bigoplus_{x \in D} T_{x_-}^* C \otimes T_{x_+}^* C$ is a gluing parameter of the domain such that they also satisfy the condition

$$a_i(x_-)a_i(x_+)\mu(x)^{s_i(x)} = \lambda \quad \text{for all } i \in I(x) \quad (4.47)$$

at each node $x \in D$ of the domain, where $a_i(x_{\pm})$ are the two leading coefficients (4.43) of \tilde{f} in the i 'th normal direction $i \in I(x)$ at the node $x \in D$.

Intrinsically, the gluing parameters μ in the domain are sections of the bundle

$$\bigoplus_{x \in D} L_{x_-} \otimes L_{x_+}$$

while the gluing parameter λ in the target is naturally a section of the bundle $N \otimes N^* \cong \mathbb{C}$. The condition (4.47) can be expressed as $k(x)$ conditions on the leading coefficients:

$$a_i(x_-)a_i(x_+) = \lambda \cdot \mu(x)^{-s_i(x)}$$

at each node. If we fix a small $\lambda \neq 0$, the existence of a $\mu(x) \neq 0$ satisfying these relations imposes a $2k(x) - 2$ dimensional condition on the leading coefficients, which is exactly what was missing in Lemma 4.7.

Remark 4.10 Notice that the enhanced matching conditions become linear if we take their log:

$$\log a_i^+(x) + \log a_i^-(x) = \log \lambda - s_i(x) \log \mu(x) \quad (4.48)$$

which makes the transversality of this condition easier to prove, and also hints to the connection with log geometry. Here log is the appropriate extension of the map $\log : \mathbb{C}^* \rightarrow \mathbb{R} \times S^1$ defined by $\log z = \log |z| + i \arg z$ to the intrinsic bundles in question.

Remark 4.11 There is another other way to read (4.46). It implies that if indeed the limit $f : C \rightarrow X_1$ does not have any components in $W_1 \cup V_1$ then of course all its leading coefficients $a_i(x_{\pm}) \neq 0, \infty$. Therefore for each fixed node x of C its contact multiplicities in the normal directions must be equal to each other:

$$s_i(x) = s_j(x) \text{ for all } i, j \in I(x).$$

Furthermore, the normalization $\tilde{f} : \tilde{C} \rightarrow \tilde{X}_1$ of the limit f must satisfy the *enhanced matching condition* at each node $x \in D$ i.e. the image of \tilde{f} under the enhanced evaluation map

$$\text{Ev}_{x_{\pm}} : \mathcal{M}_{s \sqcup s_{\pm}}(\tilde{X}_1, \tilde{W}_1) \rightarrow \mathbb{P}_{s(x)}(NW_{I(x_+)}) \times \mathbb{P}_{s(x)}(NW_{I(x_-)}) \quad (4.49)$$

lands in the antidiagonal

$$\Delta_{\pm} = \{ ([a_i], [a_i^{-1}]) \mid a_i \neq 0, \infty \text{ for } i \in I(x) \}$$

Recall that for each node, the two normal bundles in the target are canonically dual to each other, or more precisely that the normal directions to the singular locus W_1 come in dual pairs, and that k of these directions are indexed by $I(x_-)$ and the other k dual ones are indexed by $I(x_+)$.

Note that the limit f is well defined only up to an overall rescaling parameter $\lambda \in \mathbb{C}^*$ that acts on the level 1 of the building, and in fact the enhanced evaluation map descends to a map on the quotient

$$\text{Ev}_{\pm} : \mathcal{M}_{s \cup s_{\pm}}(\widetilde{X}_1, \widetilde{W}_1)/\mathbb{C}^* \rightarrow \prod_{x \in D} \mathbb{P}_{s(x)}(NW_{I(x_+)}) \times \mathbb{P}_{s(x)}(NW_{I(x_-)}) \quad (4.50)$$

that combines together the enhanced evaluation maps at all the nodes D of C .

Lemma 4.12 *For generic V -compatible perturbations (J, ν) , the dimension of the inverse image of the antidiagonal Δ_{\pm} under the enhanced evaluation map (4.50) is*

$$\dim \text{Ev}_{x_{\pm}}^{-1}(\Delta_{\pm}) = \dim \mathcal{M}_s(X, V) - 2.$$

Proof. By construction f has at least one nontrivial component in level 1, which means that the \mathbb{C}^* action on the level one is without fixed points. It is then straightforward to check that for generic V -compatible perturbations (J, ν) the enhanced evaluation map (4.50) is transverse to the antidiagonal (at least when assuming the domain C is stable, see Remark 2.1). The calculations in Lemma 4.7 together with the fact that the enhanced matching conditions impose an extra $2k(x) - 2$ dimensional condition for each node imply immediately the result. \square

Unfortunately, in the presence of a depth $k \geq 2$ point, we cannot rescale the target such that the limit f has no components in the singular locus W_1 . The most we can do is to make sure it has no nontrivial components there, but at the price of getting several trivial components stuck in the singular divisor. Below are a couple of simple examples that illustrate this behavior.

Example 4.13 Consider the situation of Example 3.7. Assume f is a fixed stable map into X which has two contact points x_1 and x_2 with V , both mapped into the singular locus p of X , but such that x_1 has multiplicity (1,1) while x_2 has multiplicity (1, 2) to the two local branches of V at p . This means that in local coordinates z_i around x_i on the domain and u_1, u_2 on the target around p , the map f has the expansions

$$f(z_1) = (a_{11}z_1, a_{21}z_1) \quad f(z_2) = (a_{12}z_2, a_{22}z_2^2)$$

with finite, nonzero leading coefficients a_{ij} . Now add another marked point x_0 to the domain. As either $x_0 \rightarrow x_1$ or $x_0 \rightarrow x_2$ a constant component of f is falling into p .

Let's look at the case $x_0 \rightarrow x_1$. Assume x_0 has coordinate $z_1 = \varepsilon$ so $f(x_0) = (a_{11}\varepsilon, a_{21}\varepsilon)$ and $\varepsilon \rightarrow 0$. Following the prescription of [IP2], we need to rescale the target by $\lambda = \varepsilon$ to catch the constant component falling in. So in coordinates $u_{11} = u_1/\lambda$ and $u_{21} = u_2/\lambda$ we get

$$f_{1\lambda}(z_1) = (a_{11}z_1/\lambda, a_{21}z_1/\lambda) \quad f_{2\lambda}(z_2) = (a_{12}z_2/\lambda, a_{22}z_2^2/\lambda)$$

In the domain, when we let $w_1 = z_1/\varepsilon = z_1/\lambda$ then $f_{1\lambda}$ converges to a level one nontrivial component which in the coordinates u_{11} and u_{21} has the expansion

$$f_1(w_1) = (a_{11}w_1, a_{21}w_1)$$

This component lands in \mathbb{F}_2 and contains the marked points x_0 and x_1 (with coordinates $w_1 = 1$ and respectively $w_1 = 0$), so it is the original component of f that was falling into p as $x_1 \rightarrow x_0$.

But when we rescale the target by $\lambda = \varepsilon$, the other piece of f at x_2 also gets rescaled, and limits to trivial components in \mathbb{F}_2 . If we rescale the domain by $w_{21} = z_2/\sqrt{\varepsilon}$ then $f_{2\lambda}(w_{21}) = (a_{12}w_2/\sqrt{\varepsilon}, a_{22}w_2^2)$ converges to a trivial map in the neck

$$f_{21}(w_{21}) = (\infty, a_{22}w_{21}^2)$$

while if we rescale the domain by $w_{22} = z_2/\varepsilon$ then $f_{2\lambda}(w_{22}) = (a_{12}w_{22}, a_{22}w_{22}^2/\varepsilon)$ also converges to a trivial map in the zero divisor

$$f_{21}(w_{22}) = (a_{12}w_{22}, 0)$$

Putting all these together, we see that the limit of f as $x_0 \rightarrow x_1$ consists of a map into a level 1 building, which has one component f on level zero and 3 components f_1 , f_{21} and f_{22} on level one (all mapped into \mathbb{F}_2). But only f_1 is a nontrivial component while the other two components are trivial, one of them mapped to the singular locus between \mathbb{F}_2 and \mathbb{F}_1 while the other one is mapped into the zero divisor of \mathbb{F}_2 , see Figure 3(a).

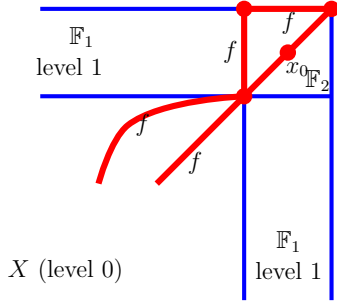
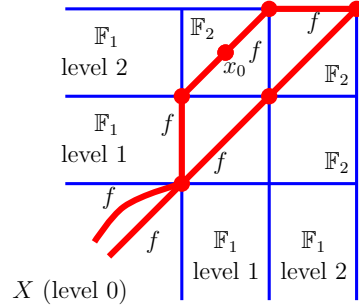


Figure 3(a): Limit as $x_0 \rightarrow x_1$



(b): Limit as $x_0 \rightarrow x_2$

One can also see what happens when $x_0 \rightarrow x_2$. Then the limit is a map into a level 2 building, which now has 5 rescaled components, only one of them nontrivial (the one containing x_0). The piece of f containing x_1 now gives rise to two trivial components f_{11} and f_{12} one on level (1,1) and the other on level (2,2) piece \mathbb{F}_2 . On the other hand, the piece of f containing x_2 gives rise to three components, the first one a trivial component in the neck between the level 1 piece \mathbb{F}_1 and \mathbb{F}_2 , the next one a nontrivial component mapped to the level (1,2) piece \mathbb{F}_2 and the last piece is a trivial component mapped into the zero divisor of the level (2,2) piece \mathbb{F}_2 , see Figure 2(b).

Note that the only nontrivial component in this case lands in level one with respect to one of the directions, but also in level two with respect to the other direction, so we have a nontrivial component in each local level.

Example 4.14 If we assumed instead that we were the case of the Example 3.6, were the two normal directions at p are globally independent, then the limit in the case $x_0 \rightarrow x_1$ would

look just the same. However, the limit when $x_0 \rightarrow x_2$ would have fewer components, as we now can rescale independently by two factors $\lambda_1 = \varepsilon$ and $\lambda_2 = \varepsilon^2$ getting a level (1,1) building. The limit then has only 3 pieces on level one, all of them mapped to \mathbb{F}_2 , but again only the piece containing x_0 is nontrivial. The other two pieces come from rescaling the piece of f containing x_1 so they are both trivial, the first one in the neck between \mathbb{F}_2 and the level (1, 0) piece \mathbb{F}_1 while the other component lands in the zero divisor of the level (1,1) piece \mathbb{F}_2 .

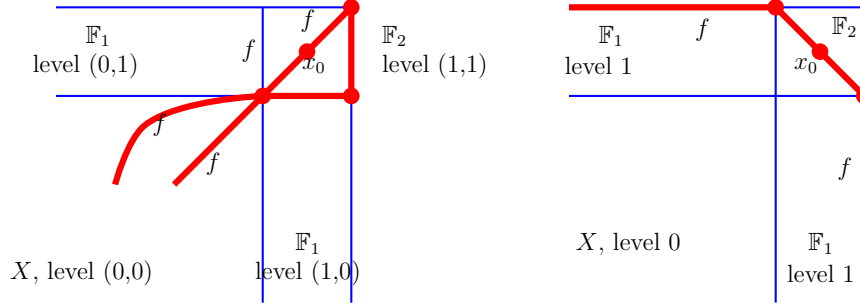


Figure 4(a) Example 4.14

(b) Example 4.15

Example 4.15 Finally, consider the case when V is a union of the first two coordinate lines in \mathbb{P}^2 and let the stable maps $f_\varepsilon : \mathbb{P}^1 \rightarrow X$ defined in homogenous coordinates by $f_\varepsilon(z) = [\varepsilon z, \varepsilon z^{-1}, 1]$, all containing a marked point x with coordinate $z = 1$. Then as $\varepsilon \rightarrow 0$ the image of marked point $f(x) = [\varepsilon, \varepsilon, 1]$ falls into $p = [0, 0, 1]$. Rescaling the target around p to prevent this gives rise to a level 1 building. The limit f has now three components, all on level 1. Out of these, only one is nontrivial and is mapped into \mathbb{F}_2 (the one containing x) while the other two are trivial, each mapped into the zero section of a different \mathbb{F}_1 piece.

The examples above show that we cannot avoid trivial components in the neck or in the zero divisor when $k(x) \geq 2$. The trivial components are uniquely determined by the behaviour of the rest of the curve and the rescaling parameter, and are there only to make the limit continuous, such that the maps converge in Hausdorff distance to their limit. The trivial components satisfy only some partial version of the matching conditions, because some of their leading coefficients (but not all!) are either zero or infinity.

5 The general limit of a sequence of maps

Now we are ready to describe what kind of maps appear as limits of maps in $\mathcal{M}_s(X, V)$. To construct the limit we will inductively rescale the sequence $f_n : C_n \rightarrow X$ to prevent (nontrivial) components from sinking into V , as described in the previous section. The limit therefore will be a (J_0, ν_0) -holomorphic map $f : C \rightarrow X_m$ in a level m building, with no *nontrivial* components in the total divisor D_m , and which will satisfy a certain enhanced matching condition at depth $k \geq 2$ points. Note however that unlike the case of [IP1], in the limit there might be some trivial components lying in the total divisor, and this is something that cannot be avoided when $k \geq 2$, see Examples 4.13 and 4.14 above. Also, the matching conditions are much more involved in this case, and are trickier to state because of the presence of these trivial curves. We start with the following:

Proposition 5.1 Consider $\{f_n : C_n \rightarrow X\}$ a sequence of maps in $\mathcal{M}_s(X, V)$. Then there exists an $m \geq 0$ and a sequence of rescaling parameters $\lambda_n = (\lambda_{n,1}, \dots, \lambda_{n,m})$ such that after passing to a subsequence, the rescaled sequence $R_{\lambda_n} f_n$ has a continuous limit $f : C \rightarrow X_m$ into a level m building with the following properties:

(a) f is a refinement of the stable map limit $f_0 : C_0 \rightarrow X$:

$$\begin{array}{ccc} C & \xrightarrow{f} & X_m \\ \downarrow \text{st} & & \downarrow p \\ C_0 & \xrightarrow{f_0} & X \end{array} \quad (5.1)$$

(b) f has a lift $\tilde{f} : \tilde{C} \rightarrow \tilde{X}_m$ to the resolution of X_m which comes with a full contact information to the total divisor D_m for each nontrivial component, and a possibly partial one to the higher depth stratum for each trivial component;

(c) for any intermediate curve $C \rightarrow C' \rightarrow C_0$ all the contact points of \tilde{f} descend to special points of C' , and moreover

- $s(x_-) = s(x_+)$ and $\varepsilon(x_-) = -\varepsilon(x_+)$ for each node $x_- = x_+$ of C' ;
- $s(x) = s(x_n)$ and $\varepsilon(x) = \varepsilon(x_n)$ for each marked point $x \in C'$ which is the limit of marked points x_n of C_n

(whenever both sides are defined). Furthermore, C is obtained from C_0 by inserting strings of trivial components B_x (broken cylinders) either between two branches x_{\pm} of a node x of C_0 or else at a marked point x of C_0 ;

(d) f is relatively stable, that is for each $l \geq 1$, f has at least one nontrivial component on level l in some (local) direction.

The limit f is unique up to the $(\mathbb{C}^*)^m$ action that rescales on X_m (described in Remark 3.16).

Proof. The existence of the refined limit follows immediately by iterating the rescaling procedure of Proposition 4.2, applied at each step l to the previously rescaled sequence. Because both the topological type of the domain as well as the homology class of the image of f_n is fixed (being part of s according to our conventions), we have a uniform bound E on the energy of this sequence, and therefore after passing to a subsequence we get a limit $f_0 : C_0 \rightarrow X$ in the usual stable map compactification, which may have some components in V , but they would carry at least $\alpha_V > 0$ energy. So we can rescale once using Lemma 4.2 to get a refined limit $g_1 : C_1 \rightarrow X_1$ which has at least $\alpha_V/2$ energy in level one and fewer nontrivial components in the zero section. After inductively rescaling around the new zero section at most $100E/\alpha_V$, this process terminates, constructing a limit with properties (a) and (d) and which furthermore has no more nontrivial components in the zero divisor.

Note that as a result of iterating the rescaling procedure of Proposition 4.2, we can only arrange that for each $l \geq 1$ there exists a nontrivial component with l as one of its (many) local levels, but not necessarily one in the global level l , as illustrated in Figure 3(a) of Example 3.7 (see Remark 3.15 for the difference between local level and global level). This is because the annular regions (3.22) where we arrange at each step l to have energy $\alpha_V/2$ have nontrivial overlap around depth $k \geq 2$ strata.

Furthermore, the limit constructed this way has no nontrivial components in the singular divisor W_m because by construction it has only energy $\alpha_V/2$ in the upper hemisphere region of each level $l \geq 1$. This means that the components of the resolution \tilde{C} of the domain C come in two types: (i) nontrivial components, which are not mapped inside the total divisor and correspond to the components of C_0 and (ii) components which are collapsed to a point

under the two maps $C \rightarrow C_0$ and $X_m \rightarrow X$. Each special fiber of $C \rightarrow C_0$ is an unstable rational curve (bubble tree) with one or two marked points.

For each nontrivial component Σ of \tilde{C} , f has a unique lift $\tilde{f}: \Sigma \rightarrow \tilde{X}_m$ to the resolution, which has a well defined contact information along the total divisor just as described in the discussion preceding the proof of Lemma 4.3, except that we now have more than one level. In particular, each point $x \in \Sigma$ comes with both a multilevel map (3.15) and a multi-sign map (3.16). For each $i \in I^\pm(x)$, $\varepsilon_i(x) = \pm$ (respectively) and we also have a well defined contact multiplicity $s_i(x) > 0$ to the total divisor and a leading coefficient $a_i(x) \neq 0$ just as in (4.39). We also have the same expansion (4.40) for all $i \in I$, as now $I^\infty(x) = \emptyset$. The nontrivial components of C therefore combine to give a partial lift \hat{f} of f which is an element of $\mathcal{M}_s(\tilde{X}_m, \tilde{D}_m)$.

This leaves us with the special fibers of $C \rightarrow C_0$, each one an unstable rational curve (bubble tree) with one or two marked points whose image gets collapsed to a point under $X_m \rightarrow X$. Each component Σ that does not land in D_m also has a unique lift to the resolution \tilde{X}_m and a well defined contact information to D_m , just like the nontrivial components did; the components that land in D_m have several lifts to the resolution and only a partial contact information to the higher depth strata of D_m , so $I^\infty(x) \neq \emptyset$ in this case. Each lift is a stable map into a fiber $(\mathbb{P}^1)^j$ of one of the many components of \tilde{X}_m , where it has a well defined contact information to the zero and infinity divisor of this fiber.

Next, the proof of Lemma 4.3 extends to the case when we have several levels of rescaling parameters (as long as they are all nonzero), giving property (c) of the limit. \square

In the remaining part of this section, we describe the behavior of the refined limit f in a neighborhood of the fiber B_x of $C \rightarrow C_0$ over a point $x \in C_0$. As we have seen, B_x is either a point x or else it is a string of trivial components with two end points x_\pm (broken cylinder). In the later case, which can happen only when x is a special point of C_0 , we order the components Σ_r of B_x in increasing order as we move from one end to the other, and make the following definition

the **stretch** of a point $x \in C_0$ is $r(x) =$ the number of components of $B_x = \text{st}^{-1}(x)$ (5.2)

where by convention $r(x) = 0$ whenever $B_x = x$.

Properties of the limit f around B_x . Consider now any node x of C_0 with its two branches x^\pm , and let $p_\pm = \tilde{f}(x^\pm) \in \tilde{X}_m$ be its two images in the resolution of X_m , while $p = f_0(x)$ is the common image in X . By construction $f(B_x)$ lies in the fiber F_p over p of the collapsing map $X_m \rightarrow X$, where we can separately work one normal direction to V at a time. Fix any of the directions $i \in I$ indexing the branches of V at p , and let π_i be the projection onto that direction, defined on a neighborhood U_p of F_p in the semi-local model described in Remark 3.17. The target of π_i is nothing but the (global) model of the deformation of a disk D^2 in N_i which is being rescaled m times at 0; it is described in terms of the coordinates $u_{i,l}$ and $v_{i,l} = u_{i,l}^{-1}$ by

$$u_{i,l-1}v_{i,l} = \lambda_{i,l} \quad \text{for all } l = 1, \dots, m$$

for any collection $\lambda = (\lambda_{i,l})$ of small rescaling parameters. Choose also local coordinates z, w at x_\pm which then induce local coordinates on the universal curve of the domains at x (the one containing C_0 , which was assumed to be stable); the nearby curves are then described in the ball $B(x, \varepsilon)$ by

$$zw = \mu(x)$$

where intrinsically the gluing parameters μ are local coordinates at C_0 on the moduli space of stable curves. Similarly choose (global) coordinates z_j , $w_j = z_j^{-1}$ on the j 'th component \mathbb{P}^1 of B_x , where $1 \leq j \leq r(x)$, and where we set $z_0 = z$ and $w_{r(x)+1} = w$. These provide global coordinates in the neighborhood O_x of B_x obtained as the inverse image of $B(x, \varepsilon)$ under the collapsing map $C \rightarrow C_0$, in which the nearby curves are described by

$$z_{r-1}w_r = \mu(x_r) \quad \text{for all } r = 1, \dots, r(x) + 1 \quad (5.3)$$

where $\mu(x_r)$ is the gluing parameter at the r 'th node x_r of B'_x , the intersection of C with O_x . In particular

$$\mu(x) = \prod_r \mu(x_r) \quad (5.4)$$

Note that B'_x has $r(x) + 2$ components Σ_r , the first and the last are the disks about x^\pm while the remaining ones are the spherical components of B_x .

For each component Σ of B_x , f may only have a partial contact information along the singular divisor at the two points 0_Σ and ∞_Σ . In fact, in the coordinates on both the domain and target described above, $f|_\Sigma$ has an associated coefficient $a_i(0_\Sigma) = a_i^{-1}(\infty_\Sigma) \neq 0$ and a contact multiplicity $s_i(\Sigma) > 0$ for all $i \in I^\pm(\Sigma)$, see Definition 4.1. Furthermore, because we already know that we have matching contact multiplicities at each node in all directions in which both sides are defined, then $s_i(\Sigma) = s_i(x)$ for all $i \notin I^\infty(\Sigma)$. In the remaining directions f still has a coefficient $a_i(\Sigma)$ which is 0 or ∞ ; the contact multiplicity $s_i(\Sigma)$ is technically undefined, but we can define it to be $s_i(x)$ for all $i \in I$.

Consider next the restriction f^i of $\pi_i \circ f$ to B'_x ; then after collapsing all the constant components (those for which $i \notin I^\pm(\Sigma)$) it has a stable map model $f^i : B'_i \rightarrow (D^2)_m$ defined on a slightly bigger curve B'_i containing B_i . We also have a similar description of the nearby curves in terms of B'_i , but where now at each node y of B'_i the gluing parameter (5.3) is

$$\mu(y) = \prod_z \mu(z) \quad (5.5)$$

where the product is over all nodes z of C in the fiber of the collapsing map $B_x \rightarrow B_i$ at y . This formula extends (5.4) which would correspond to the collapsing map $B_x \rightarrow x$.

Lemma 5.2 *Using the notations above, assume $f : C \rightarrow X_m$ is the limit of the sequence $f_n : C_n \rightarrow X$ as in Proposition 5.1. Fix a node x of C_0 and denote by $\mu_n(x_r)$ the corresponding parameters (5.3) describing the domains C_n in a neighborhood of the fiber B_x of $C \rightarrow C_0$ over x . Finally, fix a direction $i \in I(x)$ of V at $f_0(x)$.*

If $B_i \neq x$ then the restriction of f^i to B_i is a degree $s_i(x)$ chain of trivial components in $(D^2)_m$ connecting the points $\pi_i(p_+)$ to $\pi_i(p_-)$, both of which must be on the total divisor of $(D^2)_m$. In particular for each node y of B'_i , $f^i(y_-) = f^i(y_+)$ lands in the total divisor, with the two branches of f landing on opposite sides of the divisor. Furthermore, f satisfies the following enhanced matching condition at y :

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n, l_i(y)}}{\mu_n(y)^{s_i(x)}} = a_i(y_+)a_i(y_-) \quad (5.6)$$

where $a_i(y_\pm)$ and $s_i(y_\pm) = s_i(x)$ are the two leading coefficients of f , and respectively the contact multiplicity, $l_i(y)$ is the level of $f(y_\pm)$ (equal to the largest of the two consecutive levels of the lifts $\tilde{f}(y_\pm)$), while $\mu(y)$ is defined by (5.5).

When $B_i = x$ then $f^i(x_-) = f^i(x_+)$; if this lands in the total divisor of $(D^2)_m$, then f satisfies the corresponding enhanced matching condition at x in the direction i :

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n, l_i(x)}}{\mu_n(x)^{s_i(x)}} = a_i(x_-)a_i(x_+) \quad (5.7)$$

Proof. This follows by refining the arguments in the proof of Lemma 4.3, using also the information described above. Of course we work locally in the neighborhoods O_x and U_x of x and p described above, where we can separately project onto the i 'th direction. Denote by C'_n the intersection of C_n with O_x .

Because we already know that $R_{\lambda_n} f_n$ converge to f , the projections h_n^i of their restrictions to C'_n will also converge, and the limit will be precisely $f^i : B'_i \rightarrow (D^2)_m$. In fact, h_n^i are maps from C'_n into nothing but $D_{\lambda_n}^2$, the m -times rescaled disk using the rescaling parameters $\lambda_n = (\lambda_{n,1}, \dots, \lambda_{n,m})$, and that is precisely the situation in which Lemma 5.3 of [IP2] applies to give enhanced matching conditions at each mode y of B'_i . \square

Remark 5.3 The domain C of the limit f therefore is obtained from C_0 by inserting strings of trivial spheres B_x to stretch the image curve across the levels $l_i(x^\pm)$ in a zig-zagging fashion either (a) between x_- and x_+ if x is a node of C_0 or (b) at a contact point x_- of C_0 with its respective zero section to stretch it all the way to a contact point $x_+ \in C$ that is mapped to the zero section V_m . The neck region of C_n at x is roughly equal to a trivial cylinder mapped in the fiber of the neck of the target over $f_0(p)$, which then gets further stretched, possibly several times to accommodate the rescaling done to catch all the nontrivial components of f .

More precisely, each component of B_x comes with an associated multi-level map (3.15) and its two special points come with opposite but partially defined multi-sign maps (3.16). Similarly both points $f(x_\pm)$ have an associated multi-level map and a multi-sign map (now defined everywhere) pointing into opposites sides of the singular divisor in each direction (towards each other). As we have seen in Example 4.15, $\tilde{f}(x_+)$ could be on a higher level compared to $\tilde{f}(x_-)$ in some directions, and lower level in some other directions so we denote

$$l_i^-(x) = \min_{\pm} \{l_i(f(x_\pm))\} \quad \text{and} \quad l_i^+(x) = \max_{\pm} \{l_i(f(x_\pm))\} \quad (5.8)$$

The chain of trivial components connects $f(x_-)$ and $f(x_+)$ in the fiber F_p of X_m over $p = f_0(x)$ such that the levels change by either zero or one in a monotone way in each fixed direction, and also at each step we move in at least one direction. In particular, for each direction $i \in I$, and each level $l_i^-(x) \leq l \leq l_i^+(x)$, there is precisely one node $y_{i,l}(x)$ of B'_i on level l in direction i , which lifts to two points $y_{i,l}^\pm(x) \in \tilde{C}$ at which f has a well defined contact information in direction i , together with the string of trivial components $B_{i,l}(x) \subset B_x$ of C on which f is constant in direction i and thus which are precisely all the level l components of B_x in direction i . Note that for fixed i , the complement of $\sqcup_l B_{i,l}$ consists of precisely those components of f which are non-constant when projected into the i 'th direction, one for each level $l_i^-(x) \leq l \leq l_i^+(x)$. So an equivalent way to keep track of this information is to instead record these. For each i fixed, and each level l between $l_i(x_\pm)$ there exists a unique component $\Sigma_{r_{l,i}}$ of B_x which is on level l in direction i . As i is fixed and l moves from $l_i(x_-)$ to $l_i(x_+)$, the list $r_{l,i}$ is strictly increasing according to our conventions.

Remark 5.4 Furthermore, because the right hand side of (5.6) is finite and nonzero, eliminating the intermediate coefficients $\mu(y)$ from the enhanced matching conditions a fortiori gives conditions on the relative rates of convergence of the rescaling parameters (involving the contact multiplicities), extending those of Remark 4.11. The precise formulas of which

relative rates to consider depend on the contact multiplicities $s_i(x)$ plus finite combinatorial information from B_x : for each component Σ of B_x , we need to know its multilevel map l_Σ and the directions $I^\infty(\Sigma)$ in which the coefficients of f are zero or infinity. This will determine in particular which components get collapsed when we project onto direction i and thus each string $B_{i,l}(x) \subset B_x$ on which f is a constant mapped to level l in that direction. The enhanced matching condition also imposes further restriction on this combinatorial information. For example assume $x \in C_0$ is a node such that $B_{i_1,l_1}(x) = B_{i_2,l_2}(x)$. If the levels $l_1 = l_2$ then the multiplicities $s_{i_1}(x) = s_{i_2}(x)$ must be the same, while if $s_{i_1}(x) \neq s_{i_2}(x)$ then $l_1 \neq l_2$ and the relative rate of convergence to zero of the two rescaling parameters in these two levels must be related, more precisely the two rates of convergence of $\lambda_{n,l_j}^{1/s_j}$ as $n \rightarrow 0$ are equal (as their limit is a bounded, nonzero constant involving the leading coefficients of f). This was the case in Example 4.13 (b).

Intrinsically, the relative rates of convergence of Lemma 5.2 can be reinterpreted as follows. Consider the following system of linear equations in the variables $\beta(l)$ for each level $l \geq 1$ and $\alpha(z)$ for each node z of C :

$$s_i(x) \sum_{z \in D_{i,l}(x)} \alpha(z) = \beta(l) \quad (5.9)$$

for all directions $i \in I$ and all levels $l_i^-(x) \leq l \leq l_i^+(x)$, where $D_{i,l}(x)$ is the collection of nodes of C which project to x under $C \rightarrow C_0$ and which are on level l in direction i . Note that the system (5.9) depends only on the topological type of the limit $f : C \rightarrow X_m$, and more precisely on the contact multiplicities and local levels of each node z of C in all the directions normal to V .

Corollary 5.5 *Consider the situation of Lemma 5.2. Then after passing to a further subsequence of f_n , the limit $f : C \rightarrow X_m$ satisfies the following enhanced matching conditions: for each level $l \geq 1$ and each node z of C there exist positive rational numbers $\beta(l)$, $\alpha(z) > 0$ which are solutions of (5.9) and also nonzero constants $d(l)$, $c(z) \neq 0$ such that*

$$a_i(y_{i,l}^-(x))a_i(y_{i,l}^+(x)) \left(\prod_{z \in D_{i,l}(x)} c(z) \right)^{s_i(x)} = d(l) \quad (5.10)$$

for all directions $i \in I$ and all levels $l_i^-(x) \leq l \leq l_i^+(x)$, where $y_{i,l}(x)$ is the unique level l node of B'_i .

In fact, for each node x of C_0 there are some parameters $t_{n,[x]} \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n,l}}{t_{n,[x]}^{\beta(l)}} = d(l) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mu_n(z)}{t_{n,[x]}^{\alpha(z)}} = c(z) \quad (5.11)$$

for each level l between the minimum and the maximum (global) levels of x , and respectively for each node z of C that projects to x under $C \rightarrow C_0$.

Proof. As we have seen above, the conclusion of Lemma 5.2 imposes conditions on the relative rates of convergence to zero of both the rescaling parameters $\lambda_{n,l}$ in the target, and also those of the domain $\mu_n(y)$. Denote by T the collection of levels $l \geq 1$ together with all the nodes y of any intermediate curves C' with $C \rightarrow C' \rightarrow C_0$ (including $C' = C$ or C_0). For each $p \in T$ let $t_{n,p} = \lambda_{n,l}$ if $p = l$ or respectively $t_{n,p} = \mu_n(y)$ if $p = y$, where $\mu(y)$ is as in (5.5). Introduce an equivalence relation on T as follows: $p_1 \sim p_2$ if there exists a

positive rational number α such that $t_{n,p_1}^\alpha/t_{n,p_2}$ is uniformly bounded away from zero and infinity for n large. This partitions T into equivalence classes $[p]$, each one corresponding to an independent (over \mathbb{Q}) direction of convergence to zero of these parameters. As T is finite, after passing to a further subsequence of f_n , we can arrange that all these quotients have a finite, nonzero limit. Therefore there exist some $d(l) \neq 0$ and $\beta(l) \in \mathbb{Q}_+$ for each level $l \geq 1$, and $c(y) \neq 0$ and $\alpha(y) \in \mathbb{Q}_+$ for each $y \in T$ such that:

$$\lim_{n \rightarrow \infty} \frac{\mu_n(y)}{t_{n,[y]}^{\alpha(y)}} = c(y) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\lambda_{n,l}}{t_{n,[l]}^{\beta(l)}} = d(l)$$

But note that if Σ is any component of B_x with z_1, z_2 its two nodes, and y the node coming from them after contracting Σ then (5.5) becomes $\mu(y) = \mu(z_1)\mu(z_2)$. Then the above asymptotics imply that for each fixed $x \in C_0$: (a) all the points $y \in T$ that project to x are equivalent with each other and (b) all the levels l between $l^-(x) = \min_i l_i^-(x)$ and $l^+(x) = \max_i l_i^+(x)$ are also equivalent to each other. Plugging the asymptotics above into (5.6) then implies that (a) and (b) are also equivalent, proving (5.11) and reducing (5.6) to (5.9) and (5.10). \square

Remark 5.6 Each trivial component of C that lands in D_m a priori comes with only a partially defined contact information to D_m , which enters in the equations (5.9) and (5.10). However, as we shall see below, knowing the contact information of all the nontrivial components then allows us to formally extend the contact information of the trivial components even in the directions I^∞ in which their coefficients are zero or infinity, and thus the geometric contact information is technically undefined. For example, we have already seen that we can associate to each node z of C that contracts to $x \in C_0$ a multiplicity $s_i(z) = s_i(x)$ in all directions i , which matches the geometric contact multiplicity in all the directions $i \notin I^\infty$ where that is defined.

Theorem 5.7 Consider $\{f_n : C_n \rightarrow X\}$ a sequence of maps in $\mathcal{M}_s(X, V)$. Then there is a sequence of rescaling parameters λ_n such that after passing to a subsequence, $R_{\lambda_n} f_n$ has a continuous limit $f : C \rightarrow X_m$ that has the properties (a)-(d) of Proposition 5.1, plus the following extra properties:

- (e) each trivial component (Σ, x_-, x_+) of C comes with a fixed isomorphism that identifies $T_{x_-} \Sigma$ with $T_{x_+}^* \Sigma$, together with a multiplicity $s_i(x_-) = s_i(x_+)$, two opposite signs $\varepsilon_i(x_-) = -\varepsilon_i(x_+) \neq 0$ and two dual elements $a_i(x_\pm) \in N_i^{\varepsilon_i(x_\pm)} \otimes T_{x_\pm}^* \Sigma$ for each direction $i \in I(\Sigma)$, which agree with the usual contact information to D_m in the directions in which that can be geometrically defined.
- (f) at each node $y \in C$, f satisfies the naive matching condition:

$$f(y_-) = f(y_+), \quad s(y_-) = s(y_+), \quad \varepsilon(y_-) = -\varepsilon(y_+) \quad (5.12)$$

while each marked point y of C , together with its full contact information appears as the limit of corresponding marked points y_n of C_n .

- (g) there exists a solution of the linear system of equations (5.9) in the first quadrant (i.e. $\beta(l) > 0$ for each level $l \geq 1$ and $\alpha(z) > 0$ for each node z of C);
- (h) there exist nonzero constants $c(z) \neq 0$ for each node z of C such that:

$$a_i(y_-)a_i(y_+)c(y)^{s_i(x)} = 1 \quad (5.13)$$

for each node $y_- = y_+$ of C that contracts to x in C_0 and for all directions $i \in I(x)$.

The limit f satisfying all these conditions is unique up to the action of a nontrivial subtorus T_s of $(\mathbb{C}^*)^m$ which preserves the conditions (5.13).

Proof. We first use Proposition 5.1 to obtain some limit $f : C \rightarrow X_m$, defined up to the $(\mathbb{C}^*)^m$ action on X_m , and which has all the properties described there. Fix such a representative $f : C \rightarrow X_m$ of the limit, and for each point x of C_0 choose coordinates around B_x in the domain and respectively around the fiber F_p of $X_m \rightarrow X$ over $p = f_0(x)$ as described before Lemma 5.2. Note that when $B_x \neq x$ this involves a choice of dual coordinates $w_j = z_j^{-1}$ at the two end points of the trivial component Σ_j which intrinsically corresponds to a choice of an isomorphism between the tangent space to Σ_j at one of the points and its dual at the other point.

Recall that each special fiber B_x of $C \rightarrow C_0$ was a string of trivial components with two end points x_{\pm} (broken cylinder), which occurred only when x was a special point of C_0 . We make the convention that if x is a marked point of C_0 then the end x_+ of B_x corresponds to the marked point $x \in C$ while the other end x_- is where B_x gets attached to the rest of the components of C .

Next, Lemma 5.2 implies that the limit f satisfies the naive matching conditions (f) at all the special points y of any of the intermediate curves $C \rightarrow C' \rightarrow C_0$ in all the directions $i \notin I^\infty(y)$. For each trivial component (Σ, y^-, y^+) of C that is part of the trivial string B_x with two end points x_{\pm} , and for all the directions $i \in I^\infty$ we formally set $s_i(y) = s_i(x)$ and $\varepsilon_i(y_{\pm}) = \varepsilon_i(x_{\pm}) = -\varepsilon_i(x_{\mp})$ respectively. With this choice the naive matching conditions (f) are satisfied now in all directions $i \in I$.

Corollary 5.5 then implies that after possibly passing to a further subsequence, the limit f satisfies both condition (h) and (5.10). Multiplying together several of the equations (5.10) and using the fact that each trivial component has reciprocal coefficients at the two end points we get the following relation at each node $x \in C_0$:

$$a_i(x^-)a_i(x^+)c(x)^{s_i(x)} = \prod_{l=l_i^-(x)}^{l_i^+(x)} d(l) \quad (5.14)$$

for all directions $i \in I$, where $c(x) = \prod_z c(z)$ is the product over all nodes z of C which contract to x under $C \rightarrow C_0$, and more generally for the node $x_{r-1}^- = x_r^+$ of B'_x :

$$a_i(x^-)a_i(x_r^+) \cdot \left(\prod_{j=1}^r c(x_j^+) \right)^{s_i(x)} = \prod_{l=l_i(x^-)}^{l_i(x_r^+)} d(l) \quad (5.15)$$

for all $i \notin I^\infty(x_r^+)$. But the target comes with a $(\mathbb{C}^*)^m$ action where $(\alpha_1, \dots, \alpha_m) \in (\mathbb{C}^*)^m$ acts on the level l coordinates $u_{l,i} = v_{l,i}^{-1}$ described in Remark 3.17 by mapping them to $\alpha_l \cdot u_{l,i} = (\alpha_l^{-1} \cdot v_{l,i})^{-1}$ and therefore also acts on the rescaling parameters $\lambda_l = u_{l-1,i}v_{l,i}$ by $\alpha_{l-1}\alpha_l^{-1}$. Because we already know that the two local branches of f point in opposite directions of the singular divisor at x^\pm (or equivalently the weights $\varepsilon_i(x^-) = -\varepsilon_i(x^+)$ of this action are opposite on the two branches) this means that $\alpha \in (\mathbb{C}^*)^m$ acts on the product of the leading coefficients $a_i(x^-)a_i(x^+)$ or equivalently on the relative rescaling parameter $\prod_{l=l_i^-(x)}^{l_i^+(x)} \lambda_{n,l}$ by precisely $\alpha_{l_i^-(x)-1} \alpha_{l_i^+(x)}^{-1}$. But then after acting on the rescaling parameters λ_n (or equivalently on the limit f) by

$$\alpha_l = \prod_{j=1}^l d(j) \text{ for each } l \geq 1$$

the equations (5.14) and (5.15) become respectively

$$a_i(x^-)a_i(x^+)c(x)^{s_i(x)} = 1 \quad (5.16)$$

all directions $i \in I$ and

$$a_i(x^-)a_i(x_r^+) \cdot \left(\prod_{j=0}^r c(x_r^+) \right)^{s_i(x)} = 1 \quad (5.17)$$

for all directions $i \notin I^\infty(x_r^+)$. If we define $a_i(x_r^+)$ by (5.17) even for those directions $i \in I^\infty(x_r^+)$, then this gives us a way to uniquely decorate the trivial components of the limit as required in (e) once the nonzero constants $c(z) \neq 0$ are chosen such that (5.16). With these decorations, the full enhanced matching conditions (5.13) are then satisfied at all nodes y of C in all directions $i \in I$. \square

Remark 5.8 In fact, we not only have a $(\mathbb{C}^*)^m$ action on the target, but also have a similar $(\mathbb{C}^*)^d$ action on the domain, where d is the number of nodes z of C . Therefore we have a combined $(\mathbb{C}^*)^d \times (\mathbb{C}^*)^m$ action on both the leading coefficients of f and also on the gluing parameters $\mu(z)$ of the domain and respectively λ_l of the target, which leave the enhanced matching equations (5.10) invariant. For each relatively prime integer solutions $\beta(l) > 0$, $\alpha(z) > 0$ of (5.9) we also get a \mathbb{C}^* subgroup that acts on $\mu(z)$ by $t^{\alpha(z)}$ and on λ_l by $t^{\beta(l)}$, see also (5.11). In fact, the subgroup of $(\mathbb{C}^*)^{d+l}$ that preserves the product of the leading coefficients of f at each node $y_{i,l}(x)$ is precisely described by the condition (5.9), and therefore its projection T onto \mathbb{C}^m leaves the equations (5.13) invariant.

6 The Relatively Stable Map Compactification

We are finally ready to define a relatively stable map into a level m building, which describes precisely the types of limits of maps in $\mathcal{M}_s(X, V)$ we could get after rescaling.

As we have seen in the previous sections, we will need to include trivial components that are mapped into the total divisor. Because not all their coefficients are zero or infinity, they are in fact stable maps into \tilde{X}_m , but a priori they have only a partial contact information with the total divisor D_m , technically undefined in the directions I^∞ where their coefficients are zero or infinity; there was however a way to formally extend this contact information in the remaining directions as well.

So starting with this section, we make the convention that a trivial component already comes with the extra choices in these directions to get a full (but possibly formal) contact information to D_m . More precisely, each trivial component (Σ, x^+, x^-) comes first with a fixed complex isomorphism that identifies $T_{x^+}\Sigma$ with $T_{x^-}^*\Sigma$, together with a multiplicity $s_i(x^+) = s_i(x^-)$, two opposite signs $\varepsilon_i(x_-) = -\varepsilon_i(x_+) \neq 0$ and two dual elements $a_i(x_\pm) \in N_i^{\varepsilon_i(x_\pm)} \otimes T_{x_\pm}^*\Sigma$ for each direction $i \in I$, which we require to agree with the usual contact information to D_m in the directions in which that can be geometrically defined. We will denote by

$$\mathcal{M}_s^{triv}(\tilde{X}_m, \tilde{D}_m) \quad (6.1)$$

the space of such "decorated" trivial components and include them as part of $\mathcal{M}_s(\tilde{X}_m, \tilde{D}_m)$, as they now come with a fully defined evaluation map and enhanced evaluation map defined

using the extra decorations. Several crucial differences still remain between the maps that have an actual geometric contact information along D_m and these trivial components, as we will see below.

With this convention, we are ready to make the following:

Definition 6.1 *A map from C into a level m building (X_m, V_m) is a continuous function $f : C \rightarrow X_m$ with the following properties. The map f comes with a lift $\tilde{f} : \tilde{C} \rightarrow \tilde{X}_m$ which is an element of $\mathcal{M}_s(\tilde{X}_m, \tilde{D}_m)$, and a contraction $f_0 : C_0 \rightarrow X$ partitioning the domain $\tilde{C} = \tilde{C}_0 \sqcup B$ into two types of components: (a) nontrivial components, none of which is entirely contained in D_m and (b) trivial (decorated) components that are contracted under $C \rightarrow C_0$ and $X_m \rightarrow X$. We require that at each node $y_- = y_+$ of C , f satisfies the naive matching condition:*

$$f(y_+) = f(y_-), \quad s(y_+) = s(y_-) \quad \text{and} \quad \varepsilon(y_+) = -\varepsilon(y_-)$$

while none of the marked points of C are mapped to the infinity divisor.

The naive matching condition is equivalent to the fact that \tilde{f} belongs to the inverse image of the diagonal under the usual evaluation map at pairs of marked points giving the nodes D of C :

$$\text{ev}_{s_-, s_+} : \mathcal{M}_s(\tilde{X}_m, \tilde{D}_m) \longrightarrow W_{s_-} \times W_{s_+} \quad (6.2)$$

extending (4.44). Recall that the normal direction to the singular divisor W come in dual pairs, and here s_- denotes the contact information associated to the branch x_- of \tilde{C} which includes not just the multiplicities $s_i(x_-)$ but also the levels $l_i(x_-)$, the signs $\varepsilon_i(x_-)$ and the indexing set $I(x_-)$ of the branches of the total divisor D_m which record the particular strata W_{s_-} of the singular divisor W that $f(x_-)$ belongs to (which according to our conventions is equal to X_m when x is an ordinary marked point, with empty contact multiplicity). The condition (6.2) encodes both the fact that all the contact points of \tilde{f} are special points of C , but also the fact that C has no nodes on the zero divisor V_m away from the singular divisor W . The leftover contact points x of \tilde{C} which are not nodes must be therefore mapped to a stratum of the zero divisor V_m (away from the singular divisor W) and record the contact information of f along the zero divisor.

It follows from this definition that C is obtained from C_0 by possibly inserting strings of trivial components B_x (broken cylinders) either between two branches x_{\pm} of a node x of C_0 or else at a marked point x_0 of C_0 ; the fact that the signs ε_i are opposite at each node imply that each chain B_x moves in a monotone zig-zagging fashion in the fiber of X_m over $f_0(x)$, exactly as described in Remark 5.3 (note that the level changes only in those directions in which the contact information is the geometric).

Definition 6.2 *We say that a map $f : C \rightarrow X_m$ as in Definition 6.1 satisfies the enhanced matching condition if its resolution \tilde{f} is in the inverse image of the antidiagonal Δ^{\pm} under the enhanced evaluation map at pairs of marked points y^{\pm} giving the nodes D of C :*

$$\text{Ev}_{s_-, s_+} : \mathcal{M}_s(\tilde{X}_m, \tilde{D}_m) \rightarrow \prod_{y \in D} \mathbb{P}_{s(y)}(NW_{I(y_-)}) \times \mathbb{P}_{s(x)}(NW_{I(y_+)}) \quad (6.3)$$

This condition extends (4.49), and keeps track not only of the image of $\tilde{f}(y_{\pm})$ in the singular divisor W but also on its leading coefficients $a_i(y_{\pm})$ as elements of two dual normal bundles $NW_{I(y_-)} \cong N^*W_{I(y_+)}$.

Recall that there is a $(\mathbb{C}^*)^m$ action on a level m building which rescales each level $l \geq 1$ by a factor $\lambda_l \in \mathbb{C}^*$, and which induces an action both on the space of maps into the building X_m and on their resolutions.

Definition 6.3 Consider the collection of maps $f : C \rightarrow X_m$ into a level m building (X_m, V_m) as in Definition 6.1, whose resolution $\tilde{f} \in \mathcal{M}_s(\tilde{X}_m, \tilde{D}_m)$ satisfies the enhanced matching condition and whose full topological data s is such that the set of equations (5.9) has a positive solution (in the first quadrant).

Such a map is called a relatively stable map into a level m building (X_m, V_m) if furthermore for any level $l \geq 1$, there is at least one nontrivial component of f which has l as one of its multi-levels.

Let $\overline{\mathcal{M}}_s(X, V)$ denote the collection of all relatively stable maps into X , up to the $(\mathbb{C}^*)^m$ action on the level m building.

A relatively stable map is therefore an equivalence class of maps up to both the reparametrizations of the domain and also rescalings of the target. The space of solutions of the equations (5.9) describes a subtorus $T_s \subset (\mathbb{C}^*)^m$ with the property that Ev descends to the quotient:

$$\text{Ev}_{s_-, s_+} : \mathcal{M}_s(\tilde{X}_m, \tilde{D}_m)/T_s \rightarrow \prod_{y \in D} \mathbb{P}_{s(y)}(NW_{I(y_+)}) \times \mathbb{P}_{s(x)}(NW_{I(y_-)}) \quad (6.4)$$

which otherwise may not be automatic, and thus has combinatorial implications on the topological type of the maps considered, like those in Remark 5.4.

Remark 6.4 In the case V has several components and we decide to rescale in c independent directions, then we have a $(\mathbb{C}^*)^{m_1} \times \dots \times (\mathbb{C}^*)^{m_c}$ action on a (multi)level $m = (m_1, \dots, m_c)$ building (X_m, V_m) that we will take the quotient by. A map $f : C \rightarrow X_m$ is then called *relatively stable* if each multilevel $l = (l_1, \dots, l_c)$ different from $(0, \dots, 0)$ has at least one nontrivial component.

The notion of stability therefore depends in how many independent directions we rescaled the target in, and therefore on the particular group action we are taking the quotient by. For example, a nontrivial component in the level $(1, 2)$ of a multi-directional building counts as a nontrivial component in both level 1 and also in level 2 if we regard it as part of a uni-directional building, see Example 4.13 (b). If there are several independent directions, there is always a projection (stabilization) map from the relatively map compactification $\overline{\mathcal{M}}_s(X, V)$ described above to a smaller compactification obtained by collapsing some multi-levels containing only trivial components (when regarded as independent multi-levels).

With these definitions, the results of section §5 and especially Theorem 5.7 becomes:

Theorem 6.5 Consider $\{f_n : C_n \rightarrow X\}$ a sequence of maps in $\mathcal{M}_s(X, V)$. Then there is a sequence of rescaling parameters λ_n , such that after passing to a subsequence, the rescaled sequence $R_{\lambda_n} f_n$ has a unique limit $f : C \rightarrow X_m$ which is a relatively stable map into X_m .

Lemma 4.12 also extends to a level m building to give:

Lemma 6.6 For generic V -compatible (J, ν) , the stratum of the relatively stable map compactification $\overline{\mathcal{M}}_s(X, V)$ that corresponds to maps into a level $m \geq 1$ building is codimension at least 2.

Note that in the presence of higher depth strata, the codimension of the stratum into a level m building is not necessarily equal to $2m$, as again illustrated by Example 4.13 (b). The stratum as $x_1 \rightarrow x_0$ is clearly only codimension 2, even though the limit is a map into a building with 2 levels. The reason is that the enhanced matching conditions impose further conditions on the rescaling parameters (in that case $\lambda_2 = \lambda_1$), and so the rescaling parameters are not anymore independent variables. The complex codimension of the stratum is only the number of independent rescaling parameters (or equivalently the dimension of the torus T_s).

Remark 6.7 To keep the analysis concrete, we have implicitly assumed that after collapsing back the trivial components all the domains of the maps are stable, see Remark 2.1. The perturbation ν used in the Lemma above is coming then from the universal curve $\overline{\mathcal{U}} \subset \mathbb{P}^N$, so it vanishes on all the trivial components (which by definition have unstable domain). In general, no matter what kind of other perturbations one may turn on, they should always be chosen to vanish on the trivial components, as these only play a topological role of recording certain identifications between points on various levels of the building (and can be completely ignored up to their combinatorial restrictions on the topological type of the maps f).

Remark 6.8 Note that if J is integrable near V then the weighted projective space $\mathbb{P}_s(NV)$ can be regarded as an exceptional divisor in the blow up of the target normal to V , and the enhanced evaluation map is the usual evaluation map into this exceptional divisor, relating it with the approach in Davis' thesis [Da] that worked very well in genus zero (but did not extend in higher genus). The only difference here is that this blow up is now a weighted blow up, which seems to keep better track of what happens in the limit when the multiplicities s_i are not equal, especially in higher genus. Of course, being a weighted blow up, it has singular strata, which may look problematic. Ignoring the fact that when the greatest common divisor $\gcd(s_i) \neq 1$ the quotient is nonreduced, all the truly singular strata correspond to when one of the coordinates are 0 or ∞ ; but the image of the enhanced evaluation map (6.3) avoids these strata anyway.

7 The relative GW invariant $GW(X, V)$

The upshot of the discussion in the previous two sections which culminated with Theorem 6.5 is that any sequence f_n in $\mathcal{M}_s(X, V)$, after rescaling it and passing to a subsequence has a unique limit which is a relatively stable map. This combined with Lemma 6.6 about the codimension of the boundary strata allows us extend the discussion of Sections 7 and 8 of [IP1] to the case of a normal crossings divisor. In particular, Theorem 8.1 of [IP1] in this context becomes

Theorem 7.1 *Assume V is a normal crossings divisor in X for some V -compatible pair (J, ω) . The space of relatively stable maps $\overline{\mathcal{M}}_s(X, V)$ is compact and it comes with a continuous map*

$$\text{st} \times \text{Ev} : \overline{\mathcal{M}}_s(X, V) \rightarrow \overline{\mathcal{M}}_{\chi(s), \ell(s)} \times \prod_{x \in P(s)} \mathbb{P}_{s(x)}(NV_{I(x)}) \quad (7.1)$$

For generic V -compatible (J, ν) the image of $\overline{\mathcal{M}}_s(X, V)$ under $\text{st} \times \text{Ev}$ defines a homology class $GW_s(X, V)$ in dimension

$$\dim \overline{\mathcal{M}}_s(X, V) = 2c_1(TX)A_s + (\dim X - 6)\frac{\chi_s}{2} + 2\ell(s) - 2A_s \cdot V \quad (7.2)$$

This class $GW_s(X, V)$ is independent of the perturbation ν and is invariant under smooth deformations of the pair (X, V) and of (ω, J) through V -compatible structures; it is called the GW invariant of X relative the normal crossings divisor V .

When V is smooth, Ev is nothing but the usual evaluation map ev into V_s and so combined with Example 1.8 we get the following:

Corollary 7.2 *When V is a smooth symplectic codimension 2 submanifold of X , the relative GW invariant constructed in Theorem 7.1 agrees with the usual relative GW invariant $GW(X, V)$, as defined in [IP1].*

Remark 7.3 The gluing formula of [IP2] can be also extended this case to prove that for generic V -compatible (J, ν) the local model of $\overline{\mathcal{M}}_s(X, V)$ normal to a boundary stratum is precisely described by the enhanced matching conditions i.e. all solutions of the enhanced matching conditions glue to give actual (J, ν) holomorphic solutions in X_λ . Technically speaking, the space of solutions to the enhanced matching conditions is not an orbifold at $\lambda = 0$, but rather it is a branched manifold. This was also the case when V was smooth, when the enhanced matching conditions were automatically satisfied, i.e. given any stable map into a building that satisfied the naive matching condition, and any gluing parameter $\lambda \neq 0$ of the target one could always find gluing parameters $\mu(x)$ at each node of the domain satisfying the enhanced matching conditions; in fact, there were $s(x)$ different choices for each node, thus the multiplicity in the gluing formula, and the source of the branching in the moduli space. The easy fix in that case is to include in the relatively stable map compactification also the corresponding roots of unity separating the different choices of the gluing parameters $\mu(x)$ at each node x , as explained for example in [I].

In the case when V is a normal crossings divisor, the story is similar. More precisely, the local model of the moduli space $\overline{\mathcal{M}}_s(X, V)$ near a limit point f is described by tuples (f, μ, λ) . Here $f : C \rightarrow X_m$ is a map into a level m building as in Definition 6.1 which is relatively stable, $(\lambda_1, \dots, \lambda_m) \in (N \otimes N^*)^m \cong \mathbb{C}^m$ are the rescaling parameters of the target, and $\mu \in \bigoplus_{x \in D} L_{x_+} \otimes L_{x_-}$ are all the gluing parameters of the domain C (including those on trivial components). The data (f, λ, μ) must also satisfy the *enhanced matching condition* (5.10) from Corollary 5.5 at each node x of C_0 :

$$a_i(y_{i,l}^+(x)) \cdot a_i(y_{i,l}^-(x)) \cdot \left(\prod_{z \in D_{i,l}(x)} \mu(z) \right)^{s_i(x)} = \lambda_l. \quad (7.3)$$

for all directions $i \in I(x)$ and all levels $l_i^-(x) \leq l \leq l_i^+(x)$, where $a_i(y_{i,l}^\pm)$ are the leading coefficients of f and $y_{i,l}(x)$ is the unique level l node of the projection $\pi_i(B'_x)$ in direction i .

The existence of a family $\lambda_l \rightarrow 0$, $\mu(z) \rightarrow 0$ of solutions to (7.3) is not automatic (it is equivalent to condition (g) of Theorem 5.7) and therefore imposes combinatorial conditions on the topological data s of f , like those in Remark 5.4. The locus of the equations (7.3), thought as equations in the parameters $(\lambda, \mu) \in \mathbb{C}^m \times \mathbb{C}^{\ell(s)}$ is smooth for $(\lambda, \mu) \neq 0$, but may be singular at $(\lambda, \mu) = 0$ (it is only a pseudo-manifold, or a branched manifold with several branches coming together at 0). But we can instead use a refined compactification over $(\lambda, \mu) = 0$ which is smooth. One way is to re-express this local model in terms of its link at the origin, which is smooth, as we have essentially done in the proof of Theorem 5.7 to get (5.13). As we have seen there, we can also eliminate the trivial components, reducing the equations (7.3) to the following equations:

$$a_i(x^-)a_i(x^+)\mu(x)^{s_i(x)} = \prod_{l=l_i^-(x)}^{l_i^+(x)} \lambda_l. \quad (7.4)$$

for all nodes x of C_0 and all directions $i \in I(x)$.

7.1 Further directions

The next question is how the GW invariants relative normal crossings divisors behave under degenerations. The degenerations we have in mind come in several flavors.

The first type of degeneration is one in which the target X degenerates, the simplest case of that being the degeneration of a symplectic sum into its pieces. This comes down to the symplectic sum formula proved in [IP2], but where now we also have a divisor going through the neck. Consider for example the situation

$$(X, V) = (X_1, V_1) \#_U (X_2, V_2)$$

described in Remark 1.14, which means that $X = X_1 \#_U X_2$, and simultaneously the divisor V is the symplectic sum of V_1 and V_2 along their common intersection with U . Then the relative GW of the sum (X, V) should be expressed in terms of the relative GW invariants of the pieces $(X_i, V_i \cup U)$; this type of formula allows one for example to compute the absolute GW invariants of a manifold obtained by iterating the symplectic sum construction.

But there are other types of symplectic sums/smoothings of the target that these relative GW invariants should enter. The next simplest example is either the 3-fold sum or 4-fold sum defined by Symington in [S] (see also [MS]). Both these constructions should have appropriate symplectic extensions to higher dimensions involving smoothings X_ε of a symplectic manifold X self intersecting itself along a symplectic normal crossings divisor V . The sum formula would then express the GW invariants of X_ε in terms of the relative GW invariants of (X, V) . A special case of this is what is called a stable degeneration in algebraic geometry, in which case one has a smooth fibration over a disk with smooth fiber X_ε for $\varepsilon \neq 0$ and whose central fiber X_0 has normal crossings singularities.

There is also a related question when the target X is fixed, but now the divisor V degenerates in X . The simplest case of that is the one in Example 1.6, and serves as the local model of more general deformations. For example, a slightly more general case would be when we have a family of smooth divisors V_ε degenerating to a normal crossings one V_0 , which let's assume has at most depth 2 points (i.e. its singular locus W is smooth). After blowing up W , this case can be reduced to the case of a symplectic sum of the blow up of X with a standard piece \mathbb{P}_W , constructed using the normal bundle of the singular locus W . The divisors now go through the neck of the symplectic sum, but their degeneration happens only in \mathbb{P}_W (therefore involves only local information around W). So if one can understand the degeneration locally near W , one can again use the sum formula to relate the GW invariants of (X, V_ε) to those of (X, V_0) .

The discussion in this paper should also extend to the case when the target X has orbifold singularities and the normal crossings divisor V itself, as well as its normal bundle has an orbifold structure. In this case the domains of the maps should also be allowed to have orbifold singularities. So again we have a very similar stratification of the domain and of the target V but now it has more strata depending also on the conjugacy classes of the isotropy groups; the evaluation maps will now take that into account as well. The corresponding enhanced matching condition will therefore include that information, in the form of an additional balanced condition at each node as in [AGV].

A. Appendix

Assume V is a normal crossings divisor in (X, ω, J) and that $\iota : \tilde{V} \rightarrow V$ is its resolution and $\pi : N \rightarrow \tilde{V}$ its normal bundle. In particular this means that we have an immersion $\iota : (U, \tilde{V}) \rightarrow (X, V)$ from some tubular neighborhood U of the zero section \tilde{V} of N .

The divisor V is stratified depending on how many local branches meet at a particular point. Denote by V^k the closed stratum of V where at least k local branches of V meet,

and let $\overset{\circ}{V}^k$ be the open stratum where precisely k local branches meet. Then $\overset{\circ}{V}^k$ is both ω -symplectic and J -holomorphic and its normal bundle in X is modeled locally on the direct sum of the normal bundles to each local branch of V :

$$N_{V^k,p} = \bigoplus_{q \in \iota^{-1}(p)} N_q = \bigoplus_{i \in I} N_{p_i} \quad (\text{A.1.})$$

where $\iota^{-1}(p) = \{p_i \mid i \in I\}$ indexes the k local branches of V meeting at $p \in \overset{\circ}{V}^k$. When V does not have *simple* normal crossings these local branches may globally intertwine. The global monodromy of N_{V^k} is determined then by the monodromy of the restriction

$$\iota : \iota^{-1}(\overset{\circ}{V}^k) \rightarrow \overset{\circ}{V}^k \quad (\text{A.2.})$$

which describes a degree k cover of $\overset{\circ}{V}^k$, its fibers indexing the k independent directions of N_{V^k} at p . The original map ι is not a covering over the singular locus of V^k , but it extends as a covering over the normalization \widetilde{V}^k of this stratum. The following lemma follows from the local model of a normal crossings divisor:

Lemma A.1. *The closed stratum V^k of V has a normalization \widetilde{V}^k which comes with a normal crossings divisor W^k corresponding to the inverse image of the higher depth stratum V^{k+1} .*

$$\iota_k : (\widetilde{V}^k, W^{k+1}) \rightarrow (V^k, V^{k+1}). \quad (\text{A.3.})$$

The normal bundle to V^k

$$\pi : N_{V^k} \rightarrow \widetilde{V}^k \quad (\text{A.4.})$$

is obtained as in (A.1.) from the line bundle $N \rightarrow \widetilde{V}$ and a degree k cover ι of \widetilde{V}^k which extends (A.2.). By separately compactifying each normal direction we get a $(\mathbb{P}^1)^k$ bundle

$$\pi_k : \mathbb{F}_k \rightarrow \widetilde{V}^k \quad (\text{A.5.})$$

which comes with a normal crossings divisor $D_{k,0} \cup D_{k,\infty} \cup F_k$ obtained by considering together its zero and infinity divisors plus the fiber F_k over the divisor W^{k+1} in the base. The normalizations of these divisors come naturally identified

$$\widetilde{F}_k \xrightarrow[\cong]{\rho_k} \widetilde{D_{k+1,\infty}} \xrightarrow[\cong]{} \widetilde{D_{k+1,0}} \quad (\text{A.6.})$$

and their normal bundles are canonically dual to each other

$$N_{F_k} \cong (N_{D_{k+1,\infty}})^* \cong N_{D_{k+1,0}}. \quad (\text{A.7.})$$

Proof. The local model allows us to construct the normalization \widetilde{V}^k of the closed stratum V^k as a smooth manifold, obtained by separating the branches that come together to form the next stratum V^{k+1} inside V^k , and simultaneously construct the normalization $\widetilde{W^{k+1}}$ of the corresponding divisor W^{k+1} of (A.3.). The model for their normal bundles is induced from $N \rightarrow \widetilde{V}$.

There are several slight complications when $k \geq 2$. First, there is no direct map from the resolution \widetilde{V}^k of the stratum V^k of V to the depth k stratum \widetilde{V}^k of \widetilde{V} (over which N is

defined). However the normalization \widetilde{V}^k of the depth k stratum of \widetilde{V} is the degree k cover of \widetilde{V}^k whose fiber can be still thought an indexing set for the k local branches of V meeting at p :

$$\begin{array}{ccc} \widetilde{V}^k & \xrightarrow{\iota} & \widetilde{V}^k \\ \downarrow \iota_k & & \downarrow \iota_k \\ \widetilde{V} \supset \widetilde{V}^k & \xrightarrow{\iota} & V^k \subset V \end{array} \quad (\text{A.8.})$$

Here the vertical arrows are normalization maps. Therefore the pullback $\iota_k^* N$ of normal bundle $N \rightarrow \widetilde{V}$ still induces the same description of the normal bundle of V^k : at each point $p \in \widetilde{V}^k$,

$$N_{V^k, p} = \bigoplus_{q \in \iota^{-1}(p)} (\iota_k^* N)_q = \bigoplus_{i \in I} N_{p_i} \quad (\text{A.9.})$$

where $\iota^{-1}(p) = \{p_i | i \in I\}$ is the indexing set for the k local branches of V meeting at p . Again, the cover ι may have nontrivial global monodromy which will induce a global monodromy in the normal bundle N_{V^k} of V^k .

Similarly, the normalization \widetilde{W}^k of the normal crossings divisor W^k of \widetilde{V}^{k-1} is also a degree k cover of the normalization \widetilde{V}^k of the depth k stratum of V^k :

$$\begin{array}{ccc} \widetilde{W}^k & \xrightarrow{\iota} & \widetilde{V}^k \\ \downarrow \iota_k & & \downarrow \iota_k \\ \widetilde{V}^{k-1} \supset W^k & \xrightarrow{\iota_{k-1}} & V^k \subset V^{k-1} \end{array} \quad (\text{A.10.})$$

The vertical maps are resolution maps, while the fiber of the top map corresponds to the indexing set of the k branches of V ; the bottom map is the restriction of $\iota_{k-1} : \widetilde{V}^{k-1} \rightarrow V^{k-1}$ to the corresponding divisor. This means that in particular that the upper left corners of (A.8.) and (A.10.) are the same, even though the lower left corners give two different factorizations:

$$\widetilde{W}^k = \widetilde{V}^k \xrightarrow{\iota} \widetilde{V}^k \quad (\text{A.11.})$$

The fiber of this map ι indexes the k local branches of V coming together at a point $p \in \widetilde{V}^k$.

Next, the $(\mathbb{P}^1)^k$ bundle

$$\pi_k : \mathbb{F}_k \longrightarrow \widetilde{V}^k$$

is obtained by separately compactifying each of the k normal directions to V along V^k , see (A.9.). This means that its fiber at a point $p \in \widetilde{V}^k$ is

$$\times_{i \in I} \mathbb{P}(N_{p_i} \oplus \mathbb{C}) \quad (\text{A.12.})$$

where I is an indexing set of the k local branches of V meeting at p . Globally, the \mathbb{P}^1 factors of (A.12.) may intertwine.

Finally, the fiber divisor F_k of \mathbb{F}_k is by definition the inverse image of the divisor W^{k+1} of \widetilde{V}^k . Therefore its normalization \widetilde{F}_k is precisely the $(\mathbb{P}^1)^k$ bundle over the normalization \widetilde{W}^{k+1} , whose fiber is (A.12.). Moreover, the normal bundle to F_k is the pull-back of the

normal bundle of W^{k+1} inside \widetilde{V}^k , which itself is the pullback of $N \rightarrow \widetilde{V}$ by ι_k of diagram (A.8.), see also (A.11.):

$$N_{F_k} = \pi_k^* N_{W^{k+1}} = \pi_k^* \iota_k^* N \quad (\text{A.13.})$$

On the other hand, the infinity divisor $D_{k+1,\infty}$ is by definition the divisor in \mathbb{F}_{k+1} where at least one of the $k+1$ fiber coordinates $(\mathbb{P}^1)^{k+1}$ is ∞ , so its resolution $\widetilde{D}_{k+1,\infty}$ is a $(\mathbb{P}^1)^k$ bundle; the base of this bundle is itself a bundle over \widetilde{V}^{k+1} , whose fiber consists of $k+1$ points, one for each of the $k+1$ directions of V coming together. Therefore by (A.11.), $\widetilde{D}_{k+1,\infty}$ also the $(\mathbb{P}^1)^k$ bundle over the normalization \widetilde{W}^{k+1} whose fiber is (A.12.). Furthermore, the normal bundle to the infinity divisor is dual to the normal bundle to the zero divisor and thus it is canonically identified to the corresponding pullback of $N^* \rightarrow \widetilde{V}$.

This means that we have a natural identification (A.6.) as $(\mathbb{P}^1)^k$ bundles over \widetilde{W}^{k+1} and also the corresponding duality (A.7.) of their normal bundles. \square

Example A.2. (Local Structure) One can see all these different stratifications and their resolutions in the local model, when V is the union of the n coordinate hyperplanes in \mathbb{C}^n , so its resolution \widetilde{V} consists of n disjoint planes \mathbb{C}^{n-1} . The strata V^k of V are given by the vanishing of at least k coordinates in \mathbb{C}^n , while the strata $\iota^{-1}(V^k)$ of \widetilde{V} are given by the vanishing of at least $k-1$ coordinates in each one of the n disjoint planes \mathbb{C}^{n-1} of \widetilde{V} . Therefore the resolution \widetilde{V}^k of V^k consists of $\binom{n}{k}$ planes \mathbb{C}^{n-k} , while the resolution $\widetilde{\widetilde{V}}^k$ of \widetilde{V}^k consists of $n \binom{n-1}{k-1} = k \binom{n}{k}$ such planes, so the map $\iota : \widetilde{\widetilde{V}}^k \rightarrow \widetilde{V}^k$ of (A.8.) is indeed a degree k cover, whose fiber labels the k planes of V coming together at a point $p \in \widetilde{V}^k$. Finally, the divisor W^{k+1} inside \widetilde{V}^k corresponds to the coordinate hyperplanes in \widetilde{V}^k , thus its resolution \widetilde{W}^{k+1} consists of $n \binom{n-1}{k} = (k+1) \binom{n}{k+1}$ planes \mathbb{C}^{n-k-1} , which is the same as the resolution of \widetilde{V}^{k+1} . This explains the diagram (A.10.) and the identification (A.11.).

We end this Appendix with a note about the space of V -compatible parameters (J, ν) used in this paper. First of all, $J \in \text{End}(TX)$ is an almost complex structure on X compatible with ω and the perturbation ν is a section in a bundle over the product of the universal curve \mathcal{U} and X , see Remark 2.1. Now assume V is a symplectic normal crossings divisor in X . We say that the pair (J, ν) is *V-compatible* if the following three conditions on their 1-jet along V are satisfied (cf Definition 3.2 of [IP2]):

(a) J preserves TV and $\nu^N|_V = 0$;

and for all $\xi \in N_V$, $v \in TV$ and $w \in TC$:

(b) $[(\nabla_\xi J + J \nabla_{J\xi} J)(v)]^N = [(\nabla_v J)\xi + J(\nabla_{Jv} J)\xi]^N$

(c) $[(\nabla_\xi \nu + J \nabla_{J\xi} \nu)(w)]^N = [(J \nabla_{\nu(w)} J)\xi]^N$

Here $\xi \rightarrow \xi^N$ is the orthogonal projection onto the normal bundle N_V of V ; this uses the metric defined by ω and J and hence depends on J . Equivalently, assume the model for the normal bundle of V is $N_V \rightarrow \widetilde{V}$ where $\iota : \widetilde{V} \rightarrow X$ is the model for V . Because by definition (J, ω) are adapted to the divisor, then in particular all the branches of V are symplectic and preserved by J ; this means that the corresponding metric induces a splitting $\iota^* TX = T\widetilde{V} \oplus N_V$ and so the conditions above can be understood, after pullback by ι , to take place over the normalization \widetilde{V} of V .

Remark A.3. We can always replace the perturbation ν by the path $t\nu$, $t \in [0, 1]$ which gives us a retraction to the space $\mathcal{J}_V(X)$ of V -compatible J 's, i.e. those that satisfy the first two conditions. There are also further projections

$$\mathcal{J}_V(X) \longrightarrow \mathcal{J}_V^1(TX) \longrightarrow \mathcal{J}_V(TX) \quad (\text{A.14.})$$

that send a V -compatible J on X to its 1-jet normal to V , and then to its restriction along V . In the case V is a smooth symplectic submanifold of X we know that $\mathcal{J}_V(X)$ is nonempty and contractible. But this may not be the case anymore when V is singular. Already Example 1.9 shows that the fact that $\mathcal{J}_V(TX)$ is nonempty is not automatic, so the existence of a J which preserves V became part what we mean by a symplectic normal crossings divisor V . As we explain below, this is enough to guarantee the existence of a V -compatible J , i.e. one that also satisfies the condition (b) on its 1-jet along V . The arguments also below imply then that the space of V -compatible J 's is contractible.

First of all, the fiber of the first map in (A.14.) is clearly contractible, and the arguments in the Appendix of [IP1] show that the fiber of the second map is also contractible. The target of the last map is the space of complex structures J in $TX|_V$ which preserve TV and are compatible with ω . These correspond to the space of metrics g on the bundle ι^*TX over the normalization \tilde{V} of V which satisfy the condition that they descend to the restriction of TX to each depth k stratum of V . But this space of metrics is convex, and thus $\mathcal{J}_V(TX)$ is contractible (when nonempty).

In particular, when the branches of V are symplectic and orthogonal wrt ω , one can first construct locally a metric g for which $N^\omega = N^g$ around each point on V ; these can then be globally patched together to give a compatible metric on $TX|_V$ and thus an ω compatible J on $TX|_V$. Finally, one can extend this J to a V -compatible one.

In the discussion above have assumed that the symplectic form ω is fixed. But we can similarly look at deformation space of V -compatible triples (ω, J, ν) or even allow for deformations of V , via smooth deformations of the immersion $\iota : \tilde{V} \rightarrow X$ and of its normal bundle $N \rightarrow \tilde{V}$ as long as the image stays a normal crossing divisor.

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